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Classification of the quantum deformations of the superalgebra $gl(1|1)$

L Frappat†§, V Hussin† and G Rideau‡

† Centre de Recherches Mathématiques, Université de Montréal, Canada

‡ Laboratoire de Physique Théorique et Mathématique, Université Paris VII, France

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Abstract. We present a classification of the possible quantum deformations of the supergroup $GL(1|1)$ and its Lie superalgebra $gl(1|1)$. In each case, the (super)commutation relations and the Hopf structures are explicitly computed. For each R-matrix, one finds two inequivalent co-products whether one chooses an unbraided or a braided framework while the corresponding structures are isomorphic as algebras. In the braided case, one recovers the classical algebra $gl(1|1)$ for suitable limits of the deformation parameters but this is no longer true in the unbraided case.

1. Introduction

Quantum deformations of the supergroups $GL(n|m)$ have been the subject of an increasing number of papers during recent years. They have been concerned with the treatment of finite-dimensional irreducible representations of the quantum superalgebra $U_q(gl(n|m)$ [1–6]. Differential calculus has also been developed from the consideration of R-matrices satisfying the Yang–Baxter equation (YBE) [7–9].

All these approaches deal with standard deformations, some with constant R-matrices, others with R-matrices depending on a spectral parameter. The systematic study of possible R-matrices satisfying the YBE and their corresponding quantum deformations is complicated in the general case of $GL(m|n)$.

Even the simplest case $GL(1|1)$, leads to various constant R-matrices. Indeed, a complete set of a (4×4) R-matrix which satisfies the constant YBE has been constructed [10] and is the starting point for the consideration of all possible continuous deformations of the linear group $GL(2)$ and the supergroup $GL(1|1)$. The extra conditions that have to be satisfied to this aim leads us to pick only solutions which are non-singular R-matrices and continuously related to some diagonal matrices.

It is already known that all the possible deformations of $GL(2)$ that possess a central determinant are given by the standard one [11–15] and the non-standard (or 'Jordanian') one [16, 17]. It is mentioned that they are both one-parameter deformations. Once the condition of central determinant is relaxed, it can be shown [18] that this 'Jordanian' matrix contains two parameters, the computation of the quantum algebra dual to the quantum group is much more difficult and is not known.

The quantum deformations of the group $GL(1|1)$ have until now not led to an exhaustive study. Only the standard one is well known [19–21]. It is a one-parameter deformation and

§ On leave of absence from Laboratoire de Physique Théorique ENSLAPP, Annecy, France.

has been generalized to two parameters [22, 23]. An alternative deformation has also been derived [24]. So the problem addressed here is to carry out such a study and construct deformations both of the supergroup and superalgebra structures. We notice that our approach deals with deformations of superstructures with even parameters in contrast to other recent approaches [25].

A point which is important and has already been mentioned [26] is the fact that what distinguishes the group and the supergroup deformations is that the corresponding R-matrices are continuous deformations of the identity matrix in the first case and of the superidentity matrix (i.e. diag$(1, 1, 1, -1)$) in the second case.

While the paper will be concerned by the supergroup deformations, it is useful to present the necessary definitions for the usual group $GL(2)$ and point out the differences for the supergroup $GL(1|1)$.

So, let us consider the Lie group $G = GL(2)$, its Lie algebra $\mathcal{G} = gl(2)$ with generators A, B, C and D such that

$$
\begin{aligned}
&[A, B] = -[D, B] = B \qquad [A, C] = -[D, C] = -C \qquad [B, C] = A - D \\
&[A, D] = 0
\end{aligned}
\tag{1}
$$

and $\mathcal{U}$ is the universal enveloping algebra of $\mathcal{G}$. The algebra $\mathcal{A} = \mathrm{Fun}(GL(2))$ is the associative unital algebra with generators a, b, c and d that commute

$$
[a, b] = [a, c] = [a, d] = [b, c] = [b, d] = [c, d] = 0. \tag{2}
$$

The two algebras $\mathcal{U}$ and $\mathcal{A}$ can be endowed with a Hopf structure, each element of $\mathcal{G} \subset \mathcal{U}$ being primitive for the co-multiplication Δ (i.e. $\forall\ X \in \mathcal{G},\ \Delta(X) = X \otimes 1 + 1 \otimes X$) and the co-multiplication Δ for $\mathcal{A}$ is implied by the usual matrix multiplication law: $\Delta T = T \dot{\otimes} T$ if $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, that is

$$
\begin{aligned}
\Delta a &= a \otimes a + b \otimes c \qquad \Delta b = a \otimes b + b \otimes d \\
\Delta c &= c \otimes a + d \otimes c \qquad \Delta d = c \otimes b + d \otimes d.
\end{aligned}
\tag{3}
$$

Moreover, there exists a non-degenerate bilinear form $\langle\ ,\ \rangle$ on $\mathcal{U} \times \mathcal{A}$ such that

$$
\begin{aligned}
&\langle A, a^k d^l b^m c^n \rangle = k\delta_{m0}\delta_{n0} \qquad \langle B, a^k d^l b^m c^n \rangle = \delta_{m1}\delta_{n0} \\
&\langle C, a^k d^l b^m c^n \rangle = \delta_{m0}\delta_{n1} \qquad \langle D, a^k d^l b^m c^n \rangle = l\delta_{m0}\delta_{n0}
\end{aligned}
\tag{4}
$$

where $a^k d^l b^m c^n$ is any element of a Poincaré–Birkhoff–Witt basis of $\mathcal{A}$ ($k, l, m, n \in \mathbb{N}$). Finally, the pairing $\langle\ ,\ \rangle$ satisfies

$$
\langle P_1 P_2, x \rangle = \langle m(P_1 \otimes P_2), x \rangle = \langle P_1 \otimes P_2, \Delta(x) \rangle \tag{5}
$$

and

$$
\langle \Delta(P), x \otimes y \rangle = \langle P, m(x \otimes y) \rangle = \langle P, xy \rangle \tag{6}
$$

where $P_1, P_2 \in \mathcal{U}$, $x, y \in \mathcal{A}$ and m denotes the multiplication. The relations (5) and (6) make the Hopf algebras $\mathcal{U}$ and $\mathcal{A}$ dual to each other.

These definitions may be extended to the Lie supergroup $GL(1|1)$ and the Lie superalgebra $gl(1|1)$. If we call as before its generators by A, B, C and D, then we have

$$
\begin{aligned}
&[A, B] = -[D, B] = B \qquad [A, C] = -[D, C] = -C \\
&\{B, C\} = A + D \qquad [A, D] = \{B, B\} = \{C, C\} = 0.
\end{aligned}
\tag{7}
$$

Now the algebra $\mathcal{A} = \text{Fun}(GL(1|1))$ with generators a, b, c and d satisfies

$$[a, b] = [a, c] = [a, d] = [b, c] = 0 \qquad \{b, d\} = \{c, d\} = b^2 = c^2 = 0. \tag{8}$$

Deformations of the defining relations (1) and (2) or (7) and (8) are provided by the Faddeev–Reshetikhin–Takhtajan formalism [14]. Let us define $T_1 = T \otimes I$ and $T_2 = I \otimes T$. Then the deformations are given by

$$RT_1T_2 = T_2T_1R \tag{9}$$

where R is a (4×4) matrix that satisfies the quantum Yang–Baxter equation (YBE)

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{10}$$

the last equation standing in $\text{End}(\mathbb{C}^2) \otimes \text{End}(\mathbb{C}^2) \otimes \text{End}(\mathbb{C}^2)$.

As we stated before, the (4×4) constant R-matrices satisfying the YBE have been classified in [10] and among them, the subset of non-singular R-matrices splits into two different classes:

(i) the ones continuously connected to the identity matrix diag$(1, 1, 1, 1)$, which yield to quantum deformations of the group $GL(2)$: equation (9) deforms the relations (2);

(ii) the ones continuously connected to the diagonal matrix diag$(1, 1, 1, -1)$, which yield to quantum deformations of the supergroup $GL(1|1)$: equation (9) deforms the relations (8).

In the first class, there are only two distinct deformations (one case has been discussed by Fronsdal *et al* [27] and some work [16, 17] has been done on the second case specializing in the one-parameter deformation).

Let us in the following concentrate on the second class of deformations, namely those of $gl(1|1)$.

2. Deformations of the supergroup $GL(1|1)$

The class of R-matrices satisfying the YBE and continuously connected to diag$(1, 1, 1, -1)$ consists of three inequivalent matrices

$$R_{2,2} = \begin{pmatrix} r & 0 & 0 & 0 \\ 0 & 1 & r(1-q^{-1}) & 0 \\ 0 & 0 & r^2q^{-1} & 0 \\ 0 & 0 & 0 & -rq^{-1} \end{pmatrix} \tag{11}$$

$$R_{1,2} = \begin{pmatrix} 1 & 0 & 0 & r \\ 0 & 1 & 1-q^{-1} & 0 \\ 0 & 0 & q^{-1} & 0 \\ 0 & 0 & 0 & -q^{-1} \end{pmatrix} \tag{12}$$

$$R_{1,1} = \frac{1}{r}\begin{pmatrix} s+1 & 0 & 0 & s \\ 0 & r & s & 0 \\ 0 & s & r & 0 \\ s & 0 & 0 & s-1 \end{pmatrix}. \tag{13}$$

The first two matrices are really two-parameter matrices while the last one is a one-parameter matrix, the numbers r and s being subject to the condition $r^2 - s^2 = 1$ for the matrix $R_{1,1}$.

The first case is already known, but in order to obtain complete classification, we remind ourselves of the results. The multiplication law between the generators of $\mathcal{A}_{2,2}$ is given by

$$\begin{gathered} ba - rab = 0 \qquad rca - qac = 0 \qquad bd + rdb = 0 \qquad rcd + qdc = 0 \\ ad - da + r^{-1}(q-1)bc = 0 \qquad r^2cb - qbc = 0 \qquad b^2 = c^2 = 0. \end{gathered} \tag{14}$$

Theorem 0. ([23]) The supercommutation relations for the dual algebra $\mathcal{U}_{2,2}$ which is the quantum deformation of the Lie superalgebra $gl(1|1)$ associated to the R-matrix $R_{2,2}$, are given by

$$[A, D] = 0 \qquad \{C, C\} = \{B, B\} = 0 \qquad [A, B] = -[D, B] = B$$

$$[A, C] = -[D, C] = -C \qquad \{B, C\} = \frac{q^{A+D} - 1}{q - 1}$$

and the co-multiplication structure by

$$\Delta(A) = 1 \otimes A + A \otimes 1 \qquad \Delta(B) = 1 \otimes B + B \otimes (-1)^D r^{A+D}$$

$$\Delta(D) = 1 \otimes D + D \otimes 1 \qquad \Delta(C) = 1 \otimes C + C \otimes (-1)^D \left(\frac{q}{r}\right)^{A+D}.$$

3. The case $R_{1,2}$

The multiplication law between the generators of $\mathcal{A}_{1,2}$ is obtained from (9) with $R = R_{1,2}$ as

$$\begin{aligned} &ba - ab + rqdc = 0 \qquad ca - qac = 0 \\ &bd + db - rqac = 0 \qquad cd + qdc = 0 \\ &ad - da - (1-q)bc = 0 \qquad cb - qbc = 0 \\ &(1+q)b^2 - rq(a^2 - d^2) = 0 \qquad c^2 = 0. \end{aligned} \tag{15}$$

The structure relations of the corresponding dual algebra $\mathcal{U}_{1,2}$ are obtained by computing the action of the (anti)commutators between the generators A, B, C and D of $\mathcal{U}_{1,2}$ on a Poincaré–Birkhoff–Witt basis of $\mathcal{A}_{1,2}$. Such a basis is generated by the generic elements of the type $a^k d^l b^m c^n$ where $k, l \in \mathbb{N}$ and $m, n \in \{0, 1\}$ due to the two last relations of (15). Moreover, equation (5) requires knowledge of the co-multiplication $\Delta(a^k d^l b^m c^n)$. In the case under consideration, such a computation can be done directly and we have the following lemma.

Lemma 1.

$$\Delta(a^k) = a^k \otimes a^k + \frac{q^k - 1}{q - 1} a^{k-1}b \otimes a^{k-1}c - rq^2 \frac{(q^k - 1)(q^{k-1} - 1)}{(q^2 - 1)(q - 1)} a^{k-2}dc \otimes a^{k-1}c$$

$$\Delta(d^l) = d^l \otimes d^l + \frac{q^l - 1}{q - 1} d^{l-1}c \otimes d^{l-1}b - rq^2 \frac{(q^l - 1)(q^{l-1} - 1)}{(q^2 - 1)(q - 1)} d^{l-1}c \otimes ad^{l-2}c.$$

Proof. These relations are easily proved by recurrence on k and l, using equations (3) and (15). □

One has, from the fact that Δ is an algebra homomorphism

$$\begin{aligned} \Delta(a^k d^l) = a^k d^l \otimes a^k d^l &+ q^l \frac{(q^k - 1)(q^l - 1)}{(q-1)^2} \\ &\times (a^{k-1}d^{l-1}bc \otimes a^{k-1}d^{l-1}bc - rqa^k d^{l-1}c \otimes a^{k-1}d^l c) \\ &+ \frac{q^l - 1}{q - 1}\left(a^k d^{l-1}c \otimes a^k d^{l-1}b - rq^2 \frac{q^{l-1} - 1}{q^2 - 1} a^k d^{l-1}c \otimes a^{k+1}d^{l-2}c\right) \\ &+ q^l \frac{q^k - 1}{q - 1}\left(a^{k-1}d^l b \otimes a^{k-1}d^l c - rq^{l+2} \frac{q^{k-1} - 1}{q^2 - 1} a^{k-2}d^{l+1}c \otimes a^{k-1}d^l c\right). \end{aligned} \tag{16}$$

In the same way, one can deduce that

$$\Delta(a^k d^l c) = a^k d^l c \otimes a^{k+1} d^l + a^k d^{l+1} \otimes a^k d^l c - \frac{q^l - 1}{q-1} a^k d^l c \otimes a^k d^{l-1} bc + q^{l+1} \frac{q^k - 1}{q-1} a^{k-1} d^l bc \otimes a^k d^l c \tag{17}$$

and finally that

$$\begin{aligned}\Delta(a^k d^l b) = {} & a^{k+1} d^l \otimes a^k d^l b + a^k d^l b \otimes a^k d^{l+1} - \frac{q^l - 1}{q-1} a^k d^{l-1} bc \otimes a^k d^l b \\ & + q^{l+1} \frac{q^k - 1}{q-1} a^k d^l b \otimes a^{k-1} d^l bc - r q^{l+2} \frac{q^k - 1}{q^2 - 1} a^{k-1} (a^2 - d^2) d^l \otimes a^{k-1} d^{l+1} c \\ & + r q^2 \frac{q^l - 1}{q^2 - 1} a^{k+1} d^{l-1} c \otimes a^k (a^2 - d^2) d^{l-1} \\ & - r q^2 \frac{(q^l - 1)(q^{l-1} - 1)}{(q^2 - 1)(q-1)} a^{k+1} d^{l-1} c \otimes a^{k+1} d^{l-2} bc \\ & + r q^{2l+4} \frac{(q^k - 1)(q^{k-1} - 1)}{(q^2 - 1)(q-1)} a^{k-2} d^{l+1} bc \otimes a^{k-1} d^{l+1} c \\ & + r q^2 \frac{q^{k+l+1} - 1}{(q^2 - 1)(q-1)} ((q^l - 1) a^k d^{l-1} bc \otimes a^{k+1} d^{l-1} c \\ & - q^l (q^k - 1) a^{k-1} d^{l+1} c \otimes a^{k-1} d^l bc) \\ & + r q^{l+2} \frac{(q^k - 1)(q^l - 1)}{(q^2 - 1)(q-1)} (a^k d^{l-1} bc \otimes a^{k-1} d^{l+1} c - q a^{k+1} d^{l-1} c \otimes a^{k-1} d^l bc)\end{aligned} \tag{18}$$

$$\begin{aligned}\Delta(a^k d^l bc) = {} & a^{k+1} d^l c \otimes a^{k+1} d^l b - a^k d^{l+1} b \otimes a^k d^{l+1} c + a^k d^l bc \otimes a^{k+1} d^{l+1} \\ & + a^{k+1} d^{l+1} \otimes a^k d^l bc + \frac{q^{l+1}(q^{k+1} - 1) - (q^{l+1} - 1)}{q-1} a^k d^l bc \otimes a^k d^l bc \\ & + \frac{r q^2}{q^2 - 1} (q^l - 1 - q^{l+1}(q^k - 1)) a^{k+1} d^l c \otimes a^k d^{l+1} c \\ & - r q^2 \frac{q^l - 1}{q^2 - 1} a^{k+1} d^l c \otimes a^{k+2} d^{l-1} c + r q^2 q^{l+1} \frac{q^k - 1}{q^2 - 1} a^{k-1} d^{l+2} c \otimes a^k d^{l+1} c.\end{aligned} \tag{19}$$

Now we can announce the following result.

Theorem 1. The supercommutation relations for the dual algebra $\mathcal{U}_{1,2}$ which is the quantum deformation of the Lie superalgebra $gl(1|1)$ associated to the R-matrix $R_{1,2}$, are given by (we have set $K = q^{A+D}$)

$$[A, D] = 0 \qquad \{B, C\} = \frac{K-1}{q-1} \qquad [A, B] = -[D, B] = B$$

$$[A, C] = -[D, C] = -C - \frac{2rq}{q^2 - 1}(K - q) B$$

$$\{C, C\} = \frac{-2rq}{(q^2 - 1)(q-1)} (K - 1)(K - q) \qquad \{B, B\} = 0.$$

Note that the element K is central in $\mathcal{U}_{1,2}$: $[K, X] = 0$ for $X \in \{A, B, C, D\}$.

Proof. Using the formulae (16)–(19), we see that the non-vanishing pairings are the following

$$\begin{aligned}
&\langle BC + CB, a^k d^l\rangle = \frac{q^{k+l}-1}{q-1} \\
&\langle AB - BA, a^k d^l b\rangle = -\langle DB - BD, a^k d^l b\rangle = 1 \\
&\langle AC - CA, a^k d^l c\rangle = -\langle DC - CD, a^k d^l c\rangle = -1 \\
&\langle AC - CA, a^k d^l b\rangle = -\langle DC - CD, a^k d^l b\rangle = -2rq^2\frac{q^{k+l}-1}{q^2-1} \\
&\langle C^2, a^k d^l\rangle = -rq^2\frac{(q^{k+l}-1)(q^{k+l-1}-1)}{(q^2-1)(q-1)}.
\end{aligned} \tag{20}$$

To go from formulae (20) to the equations of theorem 1, we need the following expressions

$$\begin{aligned}
&\langle A^n, a^k d^l\rangle = \langle \otimes_n A, \Delta^{(n)}(a^k d^l)\rangle = k^n \\
&\langle D^n, a^k d^l\rangle = \langle \otimes_n D, \Delta^{(n)}(a^k d^l)\rangle = l^n
\end{aligned} \tag{21}$$

obtained from the co-product (3) and the multiplication law (15). It follows immediately that

$$\langle q^A, a^k d^l\rangle = q^k \qquad \text{and} \qquad \langle q^D, a^k d^l\rangle = q^l. \tag{22}$$

Moreover, one has from equation (18) (note the shift in the exponential)

$$\langle q^{A+D-1}B, a^k d^l b\rangle = q^{k+l}. \tag{23}$$

Then by comparing equations (20), (22) and (23), we obtain the commutation relations of theorem 1. □

We now want to determine the co-multiplication structure on $\mathcal{U}_{1,2}$. The duality relation (6) applied on the generic elements $a^k d^l b^m c^n$ and $a^{k'} d^{l'} b^{m'} c^{n'}$ of the Poincaré–Birkhoff–Witt basis of $\mathcal{A}_{1,2}$ reads as

$$\begin{aligned}
\langle \Delta(P), a^k d^l b^m c^n \otimes a^{k'} d^{l'} b^{m'} c^{n'}\rangle &= \langle P, m(a^k d^l b^m c^n \otimes a^{k'} d^{l'} b^{m'} c^{n'})\rangle \\
&= \langle P, a^k d^l b^m c^n a^{k'} d^{l'} b^{m'} c^{n'}\rangle.
\end{aligned} \tag{24}$$

If $\Delta(P) = P_{(1)} \otimes P_{(2)}$ in Sweedler's notation, one has

$$\langle \Delta(P), a^k d^l b^m c^n \otimes a^{k'} d^{l'} b^{m'} c^{n'}\rangle = \langle P_{(1)}, a^k d^l b^m c^n\rangle\langle P_{(2)}, a^{k'} d^{l'} b^{m'} c^{n'}\rangle. \tag{25}$$

From knowledge of $\langle P, a^k d^l b^m c^n a^{k'} d^{l'} b^{m'} c^{n'}\rangle$ as a function of $k, l, m, n, k', l', m', n'$, and the duality relations (4), one can then deduce the possible $P_{(1)}$ and $P_{(2)}$ for any generator P of the dual algebra.

From formula (24), one has to compute the action of any generator of the algebra $\mathcal{U}_{1,2}$ on a generic element $a^k d^l b^m c^n a^{k'} d^{l'} b^{m'} c^{n'}$ where $m, n, m', n' = 0$ or 1. Using the multiplication law (15), it is possible to reorder this generic element with respect to the ordering $adbc$ given by the duality relations (4). To this aim, we need the following lemma (reordering formulae).

Lemma 2.

$$
\begin{aligned}
a^k d^l a^{k'} d^{l'} &= a^{k+k'} d^{l+l'} + q^{l'} \frac{(q^{k'}-1)(q^l-1)}{q-1} a^{k+k'-1} d^{l+l'-1} bc \\
a^k d^l a^{k'} d^{l'} b &= a^{k+k'} d^{l+l'} b + \frac{rq^2}{q^2-1} q^{l'} (q^{k'}-1)(q^l-1) a^{k+k'-1} (a^2-d^2) d^{l+l'-1} c \\
a^k d^l a^{k'} d^{l'} c &= a^{k+k'} d^{l+l'} c \\
a^k d^l b a^{k'} d^{l'} &= (-1)^{l'} a^{k+k'} d^{l+l'} b + (-1)^{l'} \frac{rq^2}{q^2-1} q^{l'} (q^{k'}-1)(q^l-1) a^{k+k'-1} (a^2-d^2) d^{l+l'-1} c \\
&\quad - (-1)^{l'} \frac{rq}{q-1} (q^{l'}-1) a^{k+k'+1} d^{l+l'-1} c - (-1)^{l'} \frac{rq^{l'+1}}{q-1} (q^{k'}-1) a^{k+k'-1} d^{l+l'+1} c \\
a^k d^l c a^{k'} d^{l'} &= (-1)^{l'} q^{k'+l'} a^{k+k'} d^{l+l'} c \\
a^k d^l b a^{k'} d^{l'} b &= (-1)^{l'} \frac{rq}{q+1} a^{k+k'} (a^2-d^2) d^{l+l'} + (-1)^{l'} rq (q^{l+l'}-1) a^{k+k'+1} d^{l+l'-1} bc \\
&\quad - (-1)^{l'} \frac{rq^2}{q-1} (q^{l'}-1) a^{k+k'+1} d^{l+l'-1} bc \\
&\quad - (-1)^{l'} \frac{rq^2}{q-1} q^{l'} (q^{k'}-1) a^{k+k'-1} d^{l+l'+1} bc \\
&\quad + (-1)^{l'} \frac{rq^3}{q^2-1} q^{l'} (q^{k'}-1)(q^l-1) a^{k+k'-1} (a^2-d^2) d^{l+l'-1} bc \\
a^k d^l b a^{k'} d^{l'} c &= (-1)^{l'} a^{k+k'} d^{l+l'} bc \\
a^k d^l c a^{k'} d^{l'} b &= (-1)^{l'} q^{k'+l'+1} a^{k+k'} d^{l+l'} bc \\
a^k d^l bc a^{k'} d^{l'} &= q^{k'+l'} a^{k+k'} d^{l+l'} bc \\
a^k d^l a^{k'} d^{l'} bc &= a^{k+k'} d^{l+l'} bc \\
a^k d^l b a^{k'} d^{l'} bc &= (-1)^{l'} \frac{rq}{q+1} a^{k+k'} (a^2-d^2) d^{l+l'} c \\
a^k d^l bc a^{k'} d^{l'} b &= \frac{rq^2}{q+1} q^{k'+l'} a^{k+k'} (a^2-d^2) d^{l+l'} c \\
a^k d^l c a^{k'} d^{l'} c &= a^k d^l c a^{k'} d^{l'} bc = a^k d^l bc a^{k'} d^{l'} c = a^k d^l bc a^{k'} d^{l'} bc = 0.
\end{aligned}
$$

Proof. The proof of the lemma is straightforward and is done by recurrence on k, l, k' and l' from equation (15). □

Theorem 2. The co-multiplication Δ of the algebra $\mathcal{U}_{1,2}$ is given by

$$
\begin{aligned}
\Delta(A) &= 1 \otimes A + A \otimes 1 + \frac{2rq}{q+1} B \otimes (-1)^D B \\
\Delta(B) &= 1 \otimes B + B \otimes (-1)^D \\
\Delta(C) &= 1 \otimes C + C \otimes (-1)^D K - \frac{rq}{q-1} B \otimes (-1)^D (K-1) \\
\Delta(D) &= 1 \otimes D + D \otimes 1 - \frac{2rq}{q+1} B \otimes (-1)^D B.
\end{aligned}
$$

Let us remark that the first and last equations of theorem 2 imply that $\Delta(K) = K \otimes K$.

Proof. It follows immediately from lemma 2 that

$$
\begin{aligned}
&\langle\Delta(A), a^kd^l\otimes a^{k'}d^{l'}\rangle = k+k' \qquad \langle\Delta(A), a^kd^lb\otimes a^{k'}d^{l'}b\rangle = (-1)^{l'}\frac{2rq}{q+1}\\
&\langle\Delta(B), a^kd^l\otimes a^{k'}d^{l'}b\rangle = 1 \qquad \langle\Delta(B), a^kd^lb\otimes a^{k'}d^{l'}\rangle = (-1)^{l'}\\
&\langle\Delta(C), a^kd^l\otimes a^{k'}d^{l'}c\rangle = 1 \qquad \langle\Delta(C), a^kd^lc\otimes a^{k'}d^{l'}\rangle = (-1)^{l'}q^{k'+l'}\\
&\langle\Delta(C), a^kd^lb\otimes a^{k'}d^{l'}\rangle = -(-1)^{l'}rq\frac{q^{k'+l'}-1}{q-1}\\
&\langle\Delta(D), a^kd^l\otimes a^{k'}d^{l'}\rangle = l+l' \qquad \langle\Delta(D), a^kd^lb\otimes a^{k'}d^{l'}b\rangle = -(-1)^{l'}\frac{2rq}{q+1}
\end{aligned}
\tag{26}
$$

and all other possible terms vanish. These last relations then imply theorem 2 by using the duality relations (4), (6) and (22). This achieves the proof. □

4. The case $R_{1,1}$

The multiplication law between the generators of $\mathcal{A}_{1,1}$ is given by relation (9) with $R = R_{1,1}$. One obtains

$$
\begin{aligned}
&ba - rab + sdc = 0 \qquad ca - rac + sdb = 0 \qquad bd + rdb - sac = 0\\
&cd + rdc - sab = 0 \qquad ad - da = 0 \qquad bc - cb = 0 \qquad b^2 = c^2 = \tfrac{1}{2}s(a^2 - d^2).
\end{aligned}
\tag{27}
$$

As before a Poincaré–Birkhoff–Witt basis of $\mathcal{A}_{1,1}$ is given by $a^kd^lb^mc^n$ where $k, l \in \mathbb{N}$ and $m, n \in \{0, 1\}$ from the last relation of (27). The computation of the co-product $\Delta(a^kd^lb^mc^n)$ where $k, l \in \mathbb{N}$ and $m, n \in \{0, 1\}$ is much more involved than in the case of $\mathcal{U}_{1,2}$ because the multiplication law (27) does not allow us to compute directly the quantities $\Delta(a^kd^lb^mc^n)$. Instead, one has to solve many recursion formulae for $\Delta(a^kd^l)$ in order to produce the desired results (see the appendix).

Theorem 3. The supercommutation relations for the dual algebra $\mathcal{U}_{1,1}$ which is the quantum deformation of the Lie superalgebra $gl(1|1)$ associated to the R-matrix $R_{1,1}$, are given by

$$
\begin{aligned}
&[A, D] = 0 \qquad \{B, C\} = \frac{1}{2}\left(\frac{K^2-1}{q^2-1} + \frac{K^{-2}-1}{q^{-2}-1}\right)\\
&[A, B] = -[D, B] = \tfrac{1}{2}B + \tfrac{1}{4}(q^{-2}K^2 + q^2K^{-2})B + \tfrac{1}{4}(q^{-2}K^2 - q^2K^{-2})C\\
&[A, C] = -[D, C] = -\tfrac{1}{2}C - \tfrac{1}{4}(q^{-2}K^2 + q^2K^{-2})C - \tfrac{1}{4}(q^{-2}K^2 - q^2K^{-2})B\\
&\{B, B\} = \{C, C\} = -\frac{1}{2}\left(\frac{K^2-1}{q^2-1} - \frac{K^{-2}-1}{q^{-2}-1}\right).
\end{aligned}
$$

Since $r^2 - s^2 = 1$, we have set for convenience $r = \frac{1}{2}(q + q^{-1})$, $s = \frac{1}{2}(q - q^{-1})$ and K is defined by $K = q^{A+D}$.

Note again that the element K is central in $\mathcal{U}_{1,1}$.

Proof. Interested readers will find details in the appendix. □

Theorem 4. The co-multiplication Δ of the algebra $\mathcal{U}_{1,1}$ is given by

$$\begin{aligned}
\Delta(A) &= 1 \otimes A + A \otimes 1 + \tfrac{1}{4}(q - q^{-1}) \\
&\quad \times ((B-C) \otimes (-1)^D q^{-1} K (B+C) + (B+C) \otimes (-1)^D q K^{-1} (B-C)) \\
\Delta(B) &= 1 \otimes B + \tfrac{1}{2}(B-C) \otimes (-1)^D K + \tfrac{1}{2}(B+C) \otimes (-1)^D K^{-1} \\
\Delta(C) &= 1 \otimes C - \tfrac{1}{2}(B-C) \otimes (-1)^D K + \tfrac{1}{2}(B+C) \otimes (-1)^D K^{-1} \\
\Delta(D) &= 1 \otimes D + D \otimes 1 - \tfrac{1}{4}(q - q^{-1}) \\
&\quad \times ((B-C) \otimes (-1)^D q^{-1} K (B+C) + (B+C) \otimes (-1)^D q K^{-1} (B-C)).
\end{aligned}$$

Again the first and last equations of theorem 4 imply that $\Delta(K) = K \otimes K$.

Proof. The proof of theorem 4 is along the same lines as the proof of theorem 2. From formula (24), one computes the action of any generator of the algebra $\mathcal{U}_{1,1}$ on a generic element $a^k d^l b^m c^n a^{k'} d^{l'} b^{m'} c^{n'}$ where $m, n, m', n' \in \{0, 1\}$. All that remains to do is to reorder this generic element with respect to the ordering $adbc$ given by the duality relations (4). The reordering formulae are much simpler than in the $\mathcal{U}_{1,2}$ case. Indeed, we have from equation (27)

$$(b \pm c) a^k d^l = (ra \mp sd)^k (\pm sa - rd)^l (b \pm c) \tag{28}$$

hence

$$\begin{aligned}
ba^k d^l &= \tfrac{1}{2}((ra - sd)^k (sa - rd)^l (b + c) + (ra + sd)^k (-sa - rd)^l (b - c)) \equiv \xi^b_{kl} \\
ca^k d^l &= \tfrac{1}{2}((ra - sd)^k (sa - rd)^l (b + c) - (ra + sd)^k (-sa - rd)^l (b - c)) \equiv \xi^c_{kl}.
\end{aligned} \tag{29}$$

Then, for any $X \in \{A, B, C, D\}$, one has

$$\langle \Delta(X), a^k d^l b^m c^n \otimes a^{k'} d^{l'} b^{m'} c^{n'} \rangle = \langle X, a^k d^l b^m c^n a^{k'} d^{l'} b^{m'} c^{n'} \rangle$$

$$= \begin{cases}
\langle X, a^{k+k'} d^{l+l'} b^{m'} c^{n'} \rangle & \text{if } m = n = 0 \\
\frac{1}{2} \langle X, a^{k+k'} d^{l+l'} \xi^b_{k'l'} b^{m'} c^{n'} \rangle & \text{if } m = 1, n = 0 \\
\frac{1}{2} \langle X, a^{k+k'} d^{l+l'} \xi^c_{k'l'} b^{m'} c^{n'} \rangle & \text{if } m = 0, n = 1 \\
\frac{1}{2} \langle X, a^{k+k'} d^{l+l'} \xi^b_{k'l'} \xi^c_{k'l'} b^{m'} c^{n'} \rangle & \text{if } m = 1, n = 1.
\end{cases} \tag{30}$$

It follows immediately from equations (29) and (30) that

$$\begin{aligned}
&\langle \Delta(A), a^k d^l \otimes a^{k'} d^{l'} \rangle = k + k' \\
&\langle \Delta(A), a^k d^l b \otimes a^{k'} d^{l'} b \rangle = \langle \Delta(A), a^k d^l c \otimes a^{k'} d^{l'} c \rangle \\
&\qquad = \tfrac{1}{4}(q - q^{-1})(-1)^{l'} (q^{k'+l'} + q^{-k'-l'}) \\
&\langle \Delta(A), a^k d^l b \otimes a^{k'} d^{l'} c \rangle = \langle \Delta(A), a^k d^l c \otimes a^{k'} d^{l'} b \rangle = -\tfrac{1}{4}(q - q^{-1})(-1)^{l'} (q^{k'+l'} - q^{-k'-l'})
\end{aligned} \tag{31a}$$

$$\begin{aligned}
&\langle \Delta(B), a^k d^l \otimes a^{k'} d^{l'} b \rangle = \langle \Delta(C), a^k d^l \otimes a^{k'} d^{l'} c \rangle = 1 \\
&\langle \Delta(B), a^k d^l b \otimes a^{k'} d^{l'} \rangle = \langle \Delta(C), a^k d^l c \otimes a^{k'} d^{l'} \rangle = \tfrac{1}{2}(-1)^{l'} (q^{k'+l'} + q^{-k'-l'}) \\
&\langle \Delta(B), a^k d^l c \otimes a^{k'} d^{l'} \rangle = \langle \Delta(C), a^k d^l b \otimes a^{k'} d^{l'} \rangle = -\tfrac{1}{2}(-1)^{l'} (q^{k'+l'} - q^{-k'-l'})
\end{aligned} \tag{31b}$$

$$\langle \Delta(D), a^k d^l \otimes a^{k'} d^{l'} \rangle = l + l'$$

$$\begin{aligned}\langle \Delta(D), a^k d^l b \otimes a^{k'} d^{l'} b \rangle &= \langle \Delta(D), a^k d^l c \otimes a^{k'} d^{l'} c \rangle \\ &= -\tfrac{1}{4}(q - q^{-1})(-1)^{l'}(q^{k'+l'} + q^{-k'-l'})\end{aligned} \tag{31c}$$

$$\langle \Delta(D), a^k d^l b \otimes a^{k'} d^{l'} c \rangle = \langle \Delta(D), a^k d^l c \otimes a^{k'} d^{l'} b \rangle = \tfrac{1}{4}(q - q^{-1})(-1)^{l'}(q^{k'+l'} - q^{-k'-l'})$$

and all other possible terms vanish. These last relations then imply theorem 4 by using the duality relations (4), (6) and (84) and (85) (for these relations, see the appendix). This achieves the proof. □

5. Braided structures

In the case of the standard deformed superalgebra $gl(1|1)$, it is known that there exist two different Hopf algebras $\mathcal{U}_q[gl(1|1)]$ and $\mathcal{U}_q[gl(1|1)]'$, the two structures being isomorphic as algebras but exhibiting two distinct Hopf structures. The former admits $gl(1|1)$ a classical limit when $q \to 1$ while such a limit does not exist for the latter, $\mathcal{U}_q[gl(1|1)]'$ being related to $\mathcal{U}_i[sl(2,\mathbb{C})]$ at a root of unity ($i^2 = -1$). A similar behaviour is proved in [23] for the $\mathcal{U}_{2,2}$ case. This is a general feature as we see later.

The existence of two inequivalent Hopf structures is related to the fact that one can choose a braided or an unbraided framework.

In the unbraided case, the deformations of relations (8) are given by (9): one finds the results stated in previous sections. As can be seen from theorems 0, 2 and 4, the corresponding deformations $\mathcal{U}_{2,2}$, $\mathcal{U}_{1,2}$ and $\mathcal{U}_{1,1}$ do not admit the classical superalgebra $gl(1|1)$ as a limit for suitable values of the deformation parameters (it is clear from the co-multiplication formulae that $(-1)^D$ does not reduce to unity in such a limit).

In the braided case, one has to introduce a 'braiding matrix' chosen here as the superidentity matrix diag$(1, 1, 1, -1)$. The braided version of (9) reads as

$$R\hat{T}_1\hat{T}_2 = \hat{T}_2\hat{T}_1 R \tag{32}$$

where $\hat{T}_i = \eta T_i$ $(i = 1, 2)$.

For $R = \mathrm{diag}(1, 1, 1, -1)$, the generators of the algebra $\mathcal{A} = \mathrm{Fun}(GL(1|1))$ now satisfy:

$$[a, b] = [a, c] = [a, d] = [b, d] = [c, d] = 0 \qquad \{b, c\} = b^2 = c^2 = 0. \tag{33}$$

(Compare this with relations (8); note that relations (33) are consistent with a natural $\mathbb{Z}_2$-gradation with the assignment of a and d even and of b and c odd).

When the R-matrix is not trivial, it is easy to compute the modified multiplication laws for the cases of $R_{2,2}$, $R_{1,2}$ and $R_{1,1}$ corresponding to the deformations of (33). One finds for $\mathcal{A}_{2,2}$:

$$\begin{aligned}&ba - rab = 0 \qquad rca - qac = 0 \\ &bd - rdb = 0 \qquad rcd - qdc = 0 \\ &ad - da + r(1 - q^{-1})cb = 0 \qquad r^2cb + qbc = 0 \\ &b^2 = c^2 = 0.\end{aligned} \tag{34a}$$

for $\mathcal{A}_{1,2}$:

$$\begin{aligned} & ba - ab + rqdc = 0 \qquad ca - qac = 0 \\ & bd - db + rqac = 0 \qquad cd - qdc = 0 \\ & ad - da + (1-q)bc = 0 \qquad cb + qbc = 0 \\ & (1+q)b^2 - rq(a^2 - d^2) = 0 \qquad c^2 = 0 \end{aligned} \tag{34b}$$

for $\mathcal{A}_{1,1}$:

$$\begin{aligned} & ba - rab + sdc = 0 \qquad ca - rac + sdb = 0 \\ & bd - rdb + sac = 0 \qquad cd - rdc + sab = 0 \\ & ad - da = 0 \qquad bc + cb = 0 \\ & b^2 = -c^2 = \tfrac{1}{2}s(a^2 - d^2). \end{aligned} \tag{34c}$$

One can convince oneself, although it requires some work, that the (super)commutation relations of the corresponding dual algebras $\mathcal{U}_{2,2}$, $\mathcal{U}_{1,2}$ and $\mathcal{U}_{1,1}$ are unchanged. In this respect the relations (34) just express the original algebras in a different basis. However, the Hopf structures are *not equivalent* to the ones presented in previous sections. One finds the following results for the co-multiplication (compare with theorems 0, 2 and 4)

for $\mathcal{U}_{2,2}$:

$$\begin{aligned} & \Delta(A) = 1 \otimes A + A \otimes 1 \qquad \Delta(B) = 1 \otimes B + B \otimes r^{A+D} \\ & \Delta(D) = 1 \otimes D + D \otimes 1 \qquad \Delta(C) = 1 \otimes C + C \otimes \left(\frac{q}{r}\right)^{A+D}. \end{aligned} \tag{35a}$$

for $\mathcal{U}_{1,2}$:

$$\begin{aligned} & \Delta(A) = 1 \otimes A + A \otimes 1 + \frac{2rq}{q+1} B \otimes B \qquad \Delta(D) = 1 \otimes D + D \otimes 1 - \frac{2rq}{q+1} B \otimes B \\ & \Delta(B) = 1 \otimes B + B \otimes 1 \qquad \Delta(C) = 1 \otimes C + C \otimes q^{A+D} - \frac{rq}{q-1} B \otimes (q^{A+D} - 1). \end{aligned} \tag{35b}$$

for $\mathcal{U}_{1,1}$:

$$\begin{aligned} \Delta(A) = {} & 1 \otimes A + A \otimes 1 + \tfrac{1}{4}(q - q^{-1}) \\ & \times((B - C) \otimes q^{A+D-1}(B + C) + (B + C) \otimes q^{-A-D+1}(B - C)) \\ \Delta(B) = {} & 1 \otimes B + \tfrac{1}{2}(B + C) \otimes q^{A+D} + \tfrac{1}{2}(B - C) \otimes q^{-A-D} \\ \Delta(C) = {} & 1 \otimes C + \tfrac{1}{2}(B + C) \otimes q^{A+D} - \tfrac{1}{2}(B - C) \otimes q^{-A-D} \\ \Delta(D) = {} & 1 \otimes D + D \otimes 1 - \tfrac{1}{4}(q - q^{-1}) \\ & \times((B - C) \otimes q^{A+D-1}(B + C) + (B + C) \otimes q^{-A-D+1}(B - C)). \end{aligned} \tag{35c}$$

Notice that the q-deformed superalgebras $\mathcal{U}_{2,2}$, $\mathcal{U}_{1,2}$ and $\mathcal{U}_{1,1}$ are now endowed with a super-Hopf structure, the co-multiplication Δ and the tensor product being $\mathbb{Z}_2$-graded, this last one satisfying

$$(X_1 \otimes Y_1)(X_2 \otimes Y_2) = (-1)^{\deg Y_1 \cdot \deg X_2}(X_1 X_2 \otimes Y_1 Y_2) \tag{36}$$

the $\mathbb{Z}_2$-gradation being defined by setting $\deg A = \deg D = 0$ and $\deg B = \deg C = 1$.

It is easy to see that the relations (34) and (35) lead to the classical $GL(1|1)$ and $gl(1|1)$, endowing the superalgebra $gl(1|1)$ with a primitive co-multiplication for suitable limits of the deformation parameters: $r, q \to 1$ for the (2,2) case, $r \to 0, q \to 1$ for the (1,2) case and $q \to 1$ (or $r \to 1, s \to 0$) for the (1,1) case. Finally, the standard deformed superalgebra $U_r[gl(1|1)]$ can be obtained by taking $q = r^2$ in the case $\mathcal{U}_{2,2}$.

6. Conclusion

Starting with a two-dimensional representation of the supergroup $GL(1|1)$ we have been able to exhibit three types of continuous deformations of both the supergroup and superalgebra structures. These are based on the R-matrix method where R satisfies the YBE. Two of the three types are new with respect to previous approaches of the same problem.

It is remarkable to notice that these results coincide, at the algebra level, with those occurring in the fermionic oscillator quantum group approach [28]. Indeed, the algebra corresponding to this fermionic oscillator appears to be isomorphic to $gl(1|1)$. We started with a three-dimensional representation of the corresponding group structure and obtained, with (9×9) R-matrices satisfying a weak version of the YBE, three non-isomorphic deformations of the superalgebra $gl(1|1)$ which can be compared with the ones obtained in this paper.

For $\mathcal{U}_{2,2}$ the correspondence is immediate and this superalgebra is related to the type III fermionic oscillator quantum superalgebra.

For $\mathcal{U}_{1,2}$, the change of basis

$$\begin{aligned} &A' = A \qquad D' = D \qquad C' = C + \left(\frac{rq}{q^2-1}(K-q) + \frac{r(p-1)}{2p^2}(K-1)\right) B \\ &B' = \frac{q-1}{p} B \end{aligned} \tag{37}$$

leads to (with $K = q^{A+D}$ and $p = \ln q$)

$$\begin{aligned} &[A', D'] = 0 \qquad \{B', C'\} = \frac{K-1}{p} \qquad [A', B'] = -[D', B'] = B' \\ &[A', C'] = -[D', C'] = -C' - \frac{r}{p}(K-1)B' \qquad \{C', C'\} = -\frac{r}{p^2}(K-1)^2 \\ &\{B', B'\} = 0 \end{aligned} \tag{38}$$

which is related to the type II fermionic oscillator quantum superalgebra.

Finally, for $\mathcal{U}_{1,1}$, we notice that a more natural basis would be (with $K = q^{A+D}$)

$$\begin{aligned} &A' = q\frac{K^2+1}{K^2+q^2} A \qquad D' = q\frac{K^2+1}{K^2+q^2} D \qquad B' = (1-q^{-2})^{1/2}(B+C) \\ &C' = (q^2-1)^{1/2}(B-C) \end{aligned} \tag{39}$$

since it gives

$$\begin{aligned} &[A', D'] = 0 \qquad \{B', C'\} = 0 \qquad [A', B'] = -[D', B'] = \tfrac{1}{2}(1+K^{-2})C' \\ &[A', C'] = -[D', C'] = \tfrac{1}{2}(1+K^2)B' \qquad \{B', B'\} = 2(1-K^{-2}) \\ &\{C', C'\} = 2(1-K^2). \end{aligned} \tag{40}$$

This last structure is easily seen to be equivalent to the type I fermionic oscillator quantum superalgebra (which clearly is a one-parameter deformation).

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Appendix. Proof of theorem 3

As stated earlier, the evaluation of the action of the generators of $\mathcal{U}_{1,1}$ on the generic elements $a^k d^l b^m c^n$ of a Poincaré–Birkhoff–Witt basis of $\mathcal{A}_{1,1}$ requires the calculation of the co-product of such an element. Let us define

$$\Delta(a^k d^l) = \sum_{i,j,i',j' \in \{0,1\}} \Delta^{kl}_{ij,i'j'} \left(\begin{pmatrix} a_1 \\ d_1 \end{pmatrix}, \begin{pmatrix} a_2 \\ d_2 \end{pmatrix} \right) b^i c^j \otimes b^{i'} c^{j'} \tag{41}$$

where the quantities $\Delta^{kl}_{ij,i'j'} \left(\begin{pmatrix} a_1 \\ d_1 \end{pmatrix}, \begin{pmatrix} a_2 \\ d_2 \end{pmatrix} \right)$ are polynomials in the formal variables $a_1 = a \otimes 1$, $a_2 = 1 \otimes a$, $d_1 = d \otimes 1$ and $d_2 = 1 \otimes d$.

From the product formula (5) and the duality relations (4), it is clear that the evaluation of the commutators between the generators of $\mathcal{U}_{1,1}$ on $a^k d^l b^m c^n$ is nothing but linear combinations of the polynomials $\Delta^{kl}_{ij,i'j'} \left(\begin{pmatrix} a_1 \\ d_1 \end{pmatrix}, \begin{pmatrix} a_2 \\ d_2 \end{pmatrix} \right)$ and their derivatives for special values of the variables a_1, a_2, d_1 and d_2. More precisely, if $P(a, d)$ is a polynomial in the variables (a, d), the duality relations (4) are equivalent to

$$\begin{aligned} &\langle A, P(a,d) b^m c^n \rangle = \frac{\partial}{\partial a} P(a,d)|_{a=d=1} \delta_{m0} \delta_{n0} \qquad \langle B, P(a,d) b^m c^n \rangle = P(a,d)|_{a=d=1} \delta_{m1} \delta_{n0} \\ &\langle C, P(a,d) b^m c^n \rangle = P(a,d)|_{a=d=1} \delta_{m0} \delta_{n1} \qquad \langle D, P(a,d) b^m c^n \rangle = \frac{\partial}{\partial d} P(a,d)|_{a=d=1} \delta_{m0} \delta_{n0}. \end{aligned} \tag{42}$$

Therefore, the evaluation of the different (anti)commutators on $a^k d^l$ gives

$$\langle BC + CB, a^k d^l \rangle = \Delta^{kl}_{10,01} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) + \Delta^{kl}_{01,10} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \tag{43a}$$

$$\langle B^2, a^k d^l \rangle = \Delta^{kl}_{10,10} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \qquad \langle C^2, a^k d^l \rangle = \Delta^{kl}_{01,01} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \tag{43b}$$

$$\langle AB - BA, a^k d^l \rangle = \frac{\partial}{\partial a_1} \Delta^{kl}_{00,10} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) - \frac{\partial}{\partial a_2} \Delta^{kl}_{10,00} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \tag{43c}$$

$$\langle AC - CA, a^k d^l \rangle = \frac{\partial}{\partial a_1} \Delta^{kl}_{00,01} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) - \frac{\partial}{\partial a_2} \Delta^{kl}_{01,00} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \tag{43d}$$

$$\langle DB - BD, a^k d^l \rangle = \frac{\partial}{\partial d_1} \Delta^{kl}_{00,10} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) - \frac{\partial}{\partial d_2} \Delta^{kl}_{10,00} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \tag{43e}$$

$$\langle DC - CD, a^k d^l \rangle = \frac{\partial}{\partial d_1} \Delta^{kl}_{00,01} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) - \frac{\partial}{\partial d_2} \Delta^{kl}_{01,00} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \tag{43f}$$

$$\langle AD - DA, a^k d^l \rangle = \frac{\partial^2}{\partial a_1 \partial d_2} \Delta^{kl}_{00,00} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) - \frac{\partial^2}{\partial a_2 \partial d_1} \Delta^{kl}_{00,00} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right). \tag{43g}$$

We begin the proof by showing the following lemma.

Lemma A.1.

$$\Delta^{kl}_{10,01}\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right)=\frac{1}{4}\left(\frac{q^{2k+2l}-1}{q^2-1}+\frac{q^{-2k-2l}-1}{q^{-2}-1}+2(k-l)\right)$$

$$\Delta^{kl}_{01,10}\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right)=\frac{1}{4}\left(\frac{q^{2k+2l}-1}{q^2-1}+\frac{q^{-2k-2l}-1}{q^{-2}-1}-2(k-l)\right)$$

$$\Delta^{kl}_{01,01}\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right)=\Delta^{kl}_{10,10}\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right)=-\frac{1}{4}\left(\frac{q^{2k+2l}-1}{q^2-1}-\frac{q^{-2k-2l}-1}{q^{-2}-1}\right).$$

Rewriting the multiplication law (27) in the following form

$$(b\pm c)\begin{pmatrix}a\\d\end{pmatrix}=M_\pm\begin{pmatrix}a\\d\end{pmatrix}(b\pm c)\qquad\text{where } M_\pm=\begin{pmatrix}r & \mp s\\ \pm s & -r\end{pmatrix}\tag{44}$$

it follows from equations (41) and (44) that

$$\begin{aligned}\Delta(a^{k+1}d^l)&=(a\otimes a+b\otimes c)\Delta(a^kd^l)\\&=(a\otimes a)\sum_{i,j,i',j'\in\{0,1\}}\Delta^{kl}_{ij,i'j'}\left(\begin{pmatrix}a_1\\d_1\end{pmatrix},\begin{pmatrix}a_2\\d_2\end{pmatrix}\right)b^ic^j\otimes b^{i'}c^{j'}\\&\quad+\frac{1}{4}\sum_{i,j,i',j'\in\{0,1\}}\sum_{\varepsilon_1,\varepsilon_2=\pm1}\varepsilon_2\Delta^{kl}_{ij,i'j'}\left(M_{\varepsilon_1}\begin{pmatrix}a_1\\d_1\end{pmatrix},M_{\varepsilon_2}\begin{pmatrix}a_2\\d_2\end{pmatrix}\right)\\&\quad\times(b^{i+1}c^j+\varepsilon_1b^ic^{j+1})\otimes(b^{i'+1}c^{j'}+\varepsilon_2b^{i'}c^{j'+1}).\end{aligned}\tag{45}$$

Looking at the different terms in $\Delta(a^{k+1}d^l)$, we get

$$\begin{aligned}\Delta^{k+1,l}_{00,00}\left(\begin{pmatrix}a_1\\d_1\end{pmatrix},\begin{pmatrix}a_2\\d_2\end{pmatrix}\right)&=a_1a_2\Delta^{kl}_{00,00}\left(\begin{pmatrix}a_1\\d_1\end{pmatrix},\begin{pmatrix}a_2\\d_2\end{pmatrix}\right)\\&\quad+\frac{s^2}{16}(a_1^2-d_1^2)(a_2^2-d_2^2)\sum_{\varepsilon_1,\varepsilon_2=\pm1}\left(\varepsilon_2\Delta^{kl}_{10,10}+\Delta^{kl}_{10,01}+\varepsilon_1\varepsilon_2\Delta^{kl}_{01,10}+\varepsilon_1\Delta^{kl}_{01,01}\right)\\&\quad\times\left(M_{\varepsilon_1}\begin{pmatrix}a_1\\d_1\end{pmatrix},M_{\varepsilon_2}\begin{pmatrix}a_2\\d_2\end{pmatrix}\right)\end{aligned}\tag{46a}$$

$$\begin{aligned}\Delta^{k+1,l}_{10,10}\left(\begin{pmatrix}a_1\\d_1\end{pmatrix},\begin{pmatrix}a_2\\d_2\end{pmatrix}\right)&=a_1a_2\Delta^{kl}_{10,10}\left(\begin{pmatrix}a_1\\d_1\end{pmatrix},\begin{pmatrix}a_2\\d_2\end{pmatrix}\right)\\&\quad+\sum_{\varepsilon_1,\varepsilon_2=\pm1}\left(\frac{1}{4}\varepsilon_2\Delta^{kl}_{00,00}+\frac{s}{8}(a_1^2-d_1^2)\varepsilon_1\varepsilon_2\Delta^{kl}_{11,00}+\frac{s}{8}(a_2^2-d_2^2)\Delta^{kl}_{00,11}\right.\\&\quad\left.+\frac{s^2}{16}(a_1^2-d_1^2)(a_2^2-d_2^2)\varepsilon_1\Delta^{kl}_{11,11}\right)\left(M_{\varepsilon_1}\begin{pmatrix}a_1\\d_1\end{pmatrix},M_{\varepsilon_2}\begin{pmatrix}a_2\\d_2\end{pmatrix}\right)\end{aligned}\tag{46b}$$

$$\begin{aligned}\Delta^{k+1,l}_{01,01}\left(\begin{pmatrix}a_1\\d_1\end{pmatrix},\begin{pmatrix}a_2\\d_2\end{pmatrix}\right)&=a_1a_2\Delta^{kl}_{01,01}\left(\begin{pmatrix}a_1\\d_1\end{pmatrix},\begin{pmatrix}a_2\\d_2\end{pmatrix}\right)\\&\quad+\sum_{\varepsilon_1,\varepsilon_2=\pm1}\left(\frac{1}{4}\varepsilon_1\Delta^{kl}_{00,00}+\frac{s}{8}(a_2^2-d_2^2)\varepsilon_1\varepsilon_2\Delta^{kl}_{00,11}\right.\\&\quad\left.+\frac{s}{8}(a_1^2-d_1^2)\Delta^{kl}_{11,00}+\frac{s^2}{16}(a_1^2-d_1^2)(a_2^2-d_2^2)\varepsilon_2\Delta^{kl}_{11,11}\right)\\&\quad\times\left(M_{\varepsilon_1}\begin{pmatrix}a_1\\d_1\end{pmatrix},M_{\varepsilon_2}\begin{pmatrix}a_2\\d_2\end{pmatrix}\right)\end{aligned}\tag{46c}$$

$$\Delta^{k+1,l}_{01,10}\left(\begin{pmatrix}a_1\\d_1\end{pmatrix},\begin{pmatrix}a_2\\d_2\end{pmatrix}\right)=a_1a_2\Delta^{kl}_{01,10}\left(\begin{pmatrix}a_1\\d_1\end{pmatrix},\begin{pmatrix}a_2\\d_2\end{pmatrix}\right)$$

$$+\sum_{\varepsilon_1,\varepsilon_2=\pm1}\left(\frac{1}{4}\varepsilon_1\varepsilon_2\Delta^{kl}_{00,00}+\frac{s}{8}(a_1^2-d_1^2)\varepsilon_2\Delta^{kl}_{11,00}+\frac{s}{8}(a_2^2-d_2^2)\varepsilon_1\Delta^{kl}_{00,11}+\frac{s^2}{16}(a_1^2-d_1^2)(a_2^2-d_2^2)\Delta^{kl}_{11,11}\right)\left(M_{\varepsilon_1}\begin{pmatrix}a_1\\d_1\end{pmatrix},M_{\varepsilon_2}\begin{pmatrix}a_2\\d_2\end{pmatrix}\right) \quad (46d)$$

$$\Delta^{k+1,l}_{10,01}\left(\begin{pmatrix}a_1\\d_1\end{pmatrix},\begin{pmatrix}a_2\\d_2\end{pmatrix}\right)=a_1a_2\Delta^{kl}_{10,01}\left(\begin{pmatrix}a_1\\d_1\end{pmatrix},\begin{pmatrix}a_2\\d_2\end{pmatrix}\right)+\sum_{\varepsilon_1,\varepsilon_2=\pm1}\left(\frac{1}{4}\Delta^{kl}_{00,00}+\frac{s}{8}(a_1^2-d_1^2)\varepsilon_1\Delta^{kl}_{11,00}+\frac{s}{8}(a_2^2-d_2^2)\varepsilon_2\Delta^{kl}_{00,11}+\frac{s^2}{16}(a_1^2-d_1^2)(a_2^2-d_2^2)\varepsilon_1\varepsilon_2\Delta^{kl}_{11,11}\right)\left(M_{\varepsilon_1}\begin{pmatrix}a_1\\d_1\end{pmatrix},M_{\varepsilon_2}\begin{pmatrix}a_2\\d_2\end{pmatrix}\right). \quad (46e)$$

Taking now the values $a_1=a_2=d_1=d_2=1$, one has easily

$$\Delta^{k+1,l}_{10,01}\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right)=\Delta^{kl}_{10,01}\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right)+\frac{1}{4}\sum_{\varepsilon_1,\varepsilon_2=\pm1}\Delta^{kl}_{00,00}\left(q^{-\varepsilon_1}\begin{pmatrix}1\\-1\end{pmatrix},q^{-\varepsilon_2}\begin{pmatrix}1\\-1\end{pmatrix}\right) \quad (47a)$$

$$\Delta^{k+1,l}_{01,10}\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right)=\Delta^{kl}_{01,10}\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right)+\frac{1}{4}\sum_{\varepsilon_1,\varepsilon_2=\pm1}\varepsilon_1\varepsilon_2\Delta^{kl}_{00,00}\left(q^{-\varepsilon_1}\begin{pmatrix}1\\-1\end{pmatrix},q^{-\varepsilon_2}\begin{pmatrix}1\\-1\end{pmatrix}\right) \quad (47b)$$

$$\Delta^{k+1,l}_{01,01}\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right)=\Delta^{kl}_{01,01}\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right)+\frac{1}{4}\sum_{\varepsilon_1,\varepsilon_2=\pm1}\varepsilon_1\Delta^{kl}_{00,00}\left(q^{-\varepsilon_1}\begin{pmatrix}1\\-1\end{pmatrix},q^{-\varepsilon_2}\begin{pmatrix}1\\-1\end{pmatrix}\right) \quad (47c)$$

$$\Delta^{k+1,l}_{10,10}\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right)=\Delta^{kl}_{10,10}\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right)+\frac{1}{4}\sum_{\varepsilon_1,\varepsilon_2=\pm1}\varepsilon_2\Delta^{kl}_{00,00}\left(q^{-\varepsilon_1}\begin{pmatrix}1\\-1\end{pmatrix},q^{-\varepsilon_2}\begin{pmatrix}1\\-1\end{pmatrix}\right) \quad (47d)$$

$$\Delta^{k+1,l}_{00,00}\left(q^{-\varepsilon_1}\begin{pmatrix}1\\-1\end{pmatrix},q^{-\varepsilon_2}\begin{pmatrix}1\\-1\end{pmatrix}\right)=q^{-\varepsilon_1-\varepsilon_2}\Delta^{kl}_{00,00}\left(q^{-\varepsilon_1}\begin{pmatrix}1\\-1\end{pmatrix},q^{-\varepsilon_2}\begin{pmatrix}1\\-1\end{pmatrix}\right). \quad (47e)$$

The last equation (47*e*) can be solved and one finds

$$\Delta^{kl}_{00,00}\left(q^{-\varepsilon_1}\begin{pmatrix}1\\-1\end{pmatrix},q^{-\varepsilon_2}\begin{pmatrix}1\\-1\end{pmatrix}\right)=q^{-k(\varepsilon_1+\varepsilon_2)}\Delta^{0l}_{00,00}\left(q^{-\varepsilon_1}\begin{pmatrix}1\\-1\end{pmatrix},q^{-\varepsilon_2}\begin{pmatrix}1\\-1\end{pmatrix}\right). \quad (48)$$

In the same way, the recursion formula for l is given by

$$\begin{aligned}\Delta(a^kd^{l+1})&=(c\otimes b+d\otimes d)\Delta(a^kd^l)\\&=(d\otimes d)\sum_{i,j,i',j'\in\{0,1\}}\Delta^{kl}_{ij,i'j'}\left(\begin{pmatrix}a_1\\d_1\end{pmatrix},\begin{pmatrix}a_2\\d_2\end{pmatrix}\right)b^ic^j\otimes b^{i'}c^{j'}\\&\quad+\frac{1}{4}\sum_{i,j,i',j'\in\{0,1\}}\sum_{\varepsilon_1,\varepsilon_2=\pm1}\varepsilon_1\Delta^{kl}_{ij,i'j'}\left(M_{\varepsilon_1}\begin{pmatrix}a_1\\d_1\end{pmatrix},M_{\varepsilon_2}\begin{pmatrix}a_2\\d_2\end{pmatrix}\right)\\&\quad\times(b^{i+1}c^j+\varepsilon_1b^ic^{j+1})\otimes(b^{i'+1}c^{j'}+\varepsilon_2b^{i'}c^{j'+1}).\end{aligned} \quad (49)$$

Looking at the different terms in $\Delta(a^k d^{l+1})$, we obtain relations analogous to (46) that lead to

$$\Delta^{k,l+1}_{10,01}\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right)=\Delta^{kl}_{10,01}\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right)+\frac{1}{4}\sum_{\varepsilon_1,\varepsilon_2=\pm1}\varepsilon_1\varepsilon_2\Delta^{kl}_{00,00}\left(q^{-\varepsilon_1}\begin{pmatrix}1\\-1\end{pmatrix},q^{-\varepsilon_2}\begin{pmatrix}1\\-1\end{pmatrix}\right) \tag{50a}$$

$$\Delta^{k,l+1}_{01,10}\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right)=\Delta^{kl}_{01,10}\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right)+\frac{1}{4}\sum_{\varepsilon_1,\varepsilon_2=\pm1}\Delta^{kl}_{00,00}\left(q^{-\varepsilon_1}\begin{pmatrix}1\\-1\end{pmatrix},q^{-\varepsilon_2}\begin{pmatrix}1\\-1\end{pmatrix}\right) \tag{50b}$$

$$\Delta^{k,l+1}_{01,01}\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right)=\Delta^{kl}_{01,01}\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right)+\frac{1}{4}\sum_{\varepsilon_1,\varepsilon_2=\pm1}\varepsilon_2\Delta^{kl}_{00,00}\left(q^{-\varepsilon_1}\begin{pmatrix}1\\-1\end{pmatrix},q^{-\varepsilon_2}\begin{pmatrix}1\\-1\end{pmatrix}\right) \tag{50c}$$

$$\Delta^{k,l+1}_{10,10}\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right)=\Delta^{kl}_{10,10}\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right)+\frac{1}{4}\sum_{\varepsilon_1,\varepsilon_2=\pm1}\varepsilon_1\Delta^{kl}_{00,00}\left(q^{-\varepsilon_1}\begin{pmatrix}1\\-1\end{pmatrix},q^{-\varepsilon_2}\begin{pmatrix}1\\-1\end{pmatrix}\right) \tag{50d}$$

$$\Delta^{k,l+1}_{00,00}\left(q^{-\varepsilon_1}\begin{pmatrix}1\\-1\end{pmatrix},q^{-\varepsilon_2}\begin{pmatrix}1\\-1\end{pmatrix}\right)=q^{-\varepsilon_1-\varepsilon_2}\Delta^{kl}_{00,00}\left(q^{-\varepsilon_1}\begin{pmatrix}1\\-1\end{pmatrix},q^{-\varepsilon_2}\begin{pmatrix}1\\-1\end{pmatrix}\right). \tag{50e}$$

Choosing $k=0$ in equation (50*e*) and taking into account equation (48), it follows that

$$\Delta^{kl}_{00,00}\left(q^{-\varepsilon_1}\begin{pmatrix}1\\-1\end{pmatrix},q^{-\varepsilon_2}\begin{pmatrix}1\\-1\end{pmatrix}\right)=q^{-(k+l)(\varepsilon_1+\varepsilon_2)}. \tag{51}$$

Plugging this last result into equations (47*a*–*d*), one gets

$$\Delta^{k+1,l}_{10,01}\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right)=\Delta^{kl}_{10,01}\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right)+\frac{1}{4}(q^{2k+2l}+q^{-2k-2l}+2) \tag{52a}$$

$$\Delta^{k+1,l}_{01,10}\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right)=\Delta^{kl}_{01,10}\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right)+\frac{1}{4}(q^{2k+2l}+q^{-2k-2l}-2) \tag{52b}$$

$$\Delta^{k+1,l}_{01,01}\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right)=\Delta^{kl}_{01,01}\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right)-\frac{1}{4}(q^{2k+2l}-q^{-2k-2l}) \tag{52c}$$

$$\Delta^{k+1,l}_{10,10}\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right)=\Delta^{kl}_{10,10}\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right)-\frac{1}{4}(q^{2k+2l}-q^{-2k-2l}). \tag{52d}$$

Hence using equations (50*a*–*d*), we obtain the results of lemma A.1. Then from equation (43*a*) it follows that

$$\langle\{B,C\},a^k d^l\rangle=\frac{1}{2}\left(\frac{q^{2k+2l}-1}{q^2-1}+\frac{q^{-2k-2l}-1}{q^{-2}-1}\right). \tag{53}$$

Similarly equation (43*b*) leads to

$$\langle B^2,a^k d^l\rangle=\langle C^2,a^k d^l\rangle=-\frac{1}{4}\left(\frac{q^{2k+2l}-1}{q^2-1}-\frac{q^{-2k-2l}-1}{q^{-2}-1}\right). \tag{54}$$

Along the same lines, one can derive recursion relations for the polynomials $\Delta^{kl}_{ij,i'j'}$ where $i+j+i'+j'$ is odd—this corresponds to the choices $(ij,i'j')=(00,01)$, (00, 10), (01, 00), (10, 00), (11, 10), (11, 01), (10, 11) and (11, 11). One has the following two lemmas.

Lemma A.2. If $i+j+i'+j'$ is odd, the polynomials $\Delta^{kl}_{ij,i'j'}\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right)$ and $\Delta^{kl}_{ij,i'j'}\left(q^{-\varepsilon_1}\begin{pmatrix}1\\-1\end{pmatrix},q^{-\varepsilon_2}\begin{pmatrix}1\\-1\end{pmatrix}\right)$ are identically vanishing.

Lemma A.3. The derivatives $\partial\Delta^{kl}_{00,01}/\partial a_1$, $\partial\Delta^{kl}_{00,10}/\partial a_1$, $\partial\Delta^{kl}_{01,00}/\partial a_2$, $\partial\Delta^{kl}_{10,00}/\partial a_2$, $\partial\Delta^{kl}_{00,01}/\partial d_1$, $\partial\Delta^{kl}_{00,10}/\partial d_1$, $\partial\Delta^{kl}_{01,00}/\partial d_2$ and $\partial\Delta^{kl}_{10,00}/\partial d_2$ taken at $\begin{pmatrix}a_1\\d_1\end{pmatrix},\begin{pmatrix}a_2\\d_2\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}$ are all zero.

Consider for example the quantity $\Delta^{kl}_{00,01}$ which satisfies the recursion relation

$$\begin{aligned}\Delta^{k+1,l}_{00,01}\left(\begin{pmatrix}a_1\\d_1\end{pmatrix},\begin{pmatrix}a_2\\d_2\end{pmatrix}\right)&=a_1a_2\Delta^{kl}_{00,01}\left(\begin{pmatrix}a_1\\d_1\end{pmatrix},\begin{pmatrix}a_2\\d_2\end{pmatrix}\right)\\&\quad+(a_1^2-d_1^2)\sum_{\varepsilon_1,\varepsilon_2=\pm1}\left(\frac{s}{8}\Delta^{kl}_{10,00}+\frac{s}{8}\varepsilon_1\Delta^{kl}_{01,00}+\frac{s^2}{16}(a_2^2-d_2^2)\varepsilon_2\Delta^{kl}_{10,11}\right.\\&\quad\left.+\frac{s^2}{16}(a_2^2-d_2^2)\varepsilon_1\varepsilon_2\Delta^{kl}_{01,11}\right)\left(M_{\varepsilon_1}\begin{pmatrix}a_1\\d_1\end{pmatrix},M_{\varepsilon_2}\begin{pmatrix}a_2\\d_2\end{pmatrix}\right)\end{aligned}\tag{55}$$

obtained from equation (45). Thus for $\begin{pmatrix}a_1\\d_1\end{pmatrix},\begin{pmatrix}a_2\\d_2\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}$, one has $\Delta^{k+1,l}_{00,01}=\Delta^{kl}_{00,01}=\Delta^{0l}_{00,01}$, while the recursion relation on l leads to $\Delta^{0,l+1}_{00,01}=\Delta^{0l}_{00,01}=\Delta^{00}_{00,01}$; hence $\Delta^{kl}_{00,01}\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right)=0$.

For $\begin{pmatrix}a_1\\d_1\end{pmatrix},\begin{pmatrix}a_2\\d_2\end{pmatrix}=q^{-\varepsilon_1}\begin{pmatrix}1\\-1\end{pmatrix},q^{-\varepsilon_2}\begin{pmatrix}1\\-1\end{pmatrix}$, one has $\Delta^{k+1,l}_{00,01}=q^{-\varepsilon_1-\varepsilon_2}\Delta^{kl}_{00,01}$ and $\Delta^{0,l+1}_{00,01}=q^{-\varepsilon_1-\varepsilon_2}\Delta^{0l}_{00,01}$; hence

$$\Delta^{kl}_{00,01}\left(q^{-\varepsilon_1}\begin{pmatrix}1\\-1\end{pmatrix},q^{-\varepsilon_2}\begin{pmatrix}1\\-1\end{pmatrix}\right)=q^{-(k+l)(\varepsilon_1+\varepsilon_2)}\Delta^{00}_{00,01}\left(q^{-\varepsilon_1}\begin{pmatrix}1\\-1\end{pmatrix},q^{-\varepsilon_2}\begin{pmatrix}1\\-1\end{pmatrix}\right)=0.$$

The same statement holds for the other cases, which proves lemma A.2.

Now taking the derivative with respect to a_1 of equation (55), one gets

$$\begin{aligned}\frac{\partial}{\partial a_1}\Delta^{k+1,l}_{00,01}\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right)&=\Delta^{kl}_{00,01}\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right)+\frac{\partial}{\partial a_1}\Delta^{kl}_{00,01}\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right)\\&\quad+\frac{s}{4}\sum_{\varepsilon_1,\varepsilon_2=\pm1}(\Delta^{kl}_{10,00}+\varepsilon_1\Delta^{kl}_{01,00})\left(q^{-\varepsilon_1}\begin{pmatrix}1\\-1\end{pmatrix},q^{-\varepsilon_2}\begin{pmatrix}1\\-1\end{pmatrix}\right)\end{aligned}\tag{56}$$

and analogous relations for the other combinations of the quadruplets $(ij,i'j')$ with $i+j+i'+j'$ odd. From lemma A.2, one has therefore

$$\frac{\partial}{\partial a_1}\Delta^{kl}_{00,01}\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right)=\frac{\partial}{\partial a_1}\Delta^{0l}_{00,01}\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right).\tag{57}$$

Repeating the procedure for the recursion relations on l, one finds that the right-hand side of (57) is zero and one concludes that

$$\frac{\partial}{\partial a_1}\Delta^{kl}_{00,01}\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right)=0.\tag{58}$$

The same statement holds for all the derivatives of the polynomials $\Delta^{kl}_{ij,i'j'}$ involved in equations (43*c*–*f*), which proves lemma A.3.

It follows then from lemmas A.2 and A.3 and equations (43*c*–*f*) that

$$\langle [A, B], a^k d^l \rangle = \langle [A, C], a^k d^l \rangle = \langle [D, B], a^k d^l \rangle = \langle [D, C], a^k d^l \rangle = 0. \tag{59}$$

It remains to evaluate $\langle AD - DA, a^k d^l \rangle$. One has from equation (46*a*)

$$\begin{aligned} \frac{\partial^2}{\partial a_1 \partial d_2} \Delta^{k+1,l}_{00,00} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) &= \frac{\partial^2}{\partial a_1 \partial d_2} \Delta^{kl}_{00,00} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) + \frac{\partial}{\partial d_2} \Delta^{kl}_{00,00} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \\ &\quad - \frac{s^2}{4} \sum_{\varepsilon_1, \varepsilon_2 = \pm 1} \left(\Delta^{kl}_{10,01} + \varepsilon_1 \Delta^{kl}_{01,01} + \varepsilon_2 \Delta^{kl}_{10,10} + \varepsilon_1 \varepsilon_2 \Delta^{kl}_{01,10} \right) \\ &\quad \times \left(q^{-\varepsilon_1} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, q^{-\varepsilon_2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) \end{aligned} \tag{60}$$

$$\begin{aligned} \frac{\partial^2}{\partial a_2 \partial d_1} \Delta^{k+1,l}_{00,00} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) &= \frac{\partial^2}{\partial a_2 \partial d_1} \Delta^{kl}_{00,00} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) + \frac{\partial}{\partial d_1} \Delta^{kl}_{00,00} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \\ &\quad - \frac{s^2}{4} \sum_{\varepsilon_1, \varepsilon_2 = \pm 1} \left(\Delta^{kl}_{10,01} + \varepsilon_1 \Delta^{kl}_{01,01} + \varepsilon_2 \Delta^{kl}_{10,10} + \varepsilon_1 \varepsilon_2 \Delta^{kl}_{01,10} \right) \\ &\quad \times \left(q^{-\varepsilon_1} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, q^{-\varepsilon_2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right). \end{aligned} \tag{61}$$

Therefore

$$\begin{aligned} \left(\frac{\partial^2}{\partial a_1 \partial d_2} - \frac{\partial^2}{\partial a_2 \partial d_1} \right) \Delta^{k+1,l}_{00,00} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) &= \left(\frac{\partial^2}{\partial a_1 \partial d_2} - \frac{\partial^2}{\partial a_2 \partial d_1} \right) \Delta^{kl}_{00,00} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \\ &\quad + \left(\frac{\partial}{\partial d_2} - \frac{\partial}{\partial d_1} \right) \Delta^{kl}_{00,00} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right). \end{aligned} \tag{62}$$

Then we use the following lemma.

Lemma A.4. One has

$$\begin{aligned} \frac{\partial}{\partial a_1} \Delta^{kl}_{00,00} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) &= \frac{\partial}{\partial a_2} \Delta^{kl}_{00,00} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = k \\ \frac{\partial}{\partial d_1} \Delta^{kl}_{00,00} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) &= \frac{\partial}{\partial d_2} \Delta^{kl}_{00,00} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = l. \end{aligned}$$

From equation (46*a*), one has

$$\frac{\partial}{\partial d_i} \Delta^{k+1,l}_{00,00} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = \frac{\partial}{\partial d_i} \Delta^{kl}_{00,00} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \tag{63}$$

and similarly

$$\frac{\partial}{\partial d_i} \Delta^{0,l+1}_{00,00} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = \Delta^{0l}_{00,00} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) + \frac{\partial}{\partial d_i} \Delta^{0l}_{00,00} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right). \tag{64}$$

Since

$$\Delta^{0,l+1}_{00,00} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \left(\frac{1}{1} \right) \right) = \Delta^{0l}_{00,00} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \qquad \text{and} \qquad \Delta^{00}_{00,00} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \left(\frac{1}{1} \right) \right) = 1$$

one obtains the last line of lemma A.4. One gets the first line by exchanging the roles of k and l.

Then lemma A.4 implies

$$\left(\frac{\partial^2}{\partial a_1\partial d_2}-\frac{\partial^2}{\partial a_2\partial d_1}\right)\Delta^{k+1,l}_{00,00}\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right)$$
$$=\left(\frac{\partial^2}{\partial a_1\partial d_2}-\frac{\partial^2}{\partial a_2\partial d_1}\right)\Delta^{kl}_{00,00}\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right). \tag{65}$$

Similarly, one gets

$$\left(\frac{\partial^2}{\partial a_1\partial d_2}-\frac{\partial^2}{\partial a_2\partial d_1}\right)\Delta^{0,l+1}_{00,00}\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right)$$
$$=\left(\frac{\partial^2}{\partial a_1\partial d_2}-\frac{\partial^2}{\partial a_2\partial d_1}\right)\Delta^{0l}_{00,00}\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right). \tag{66}$$

Hence

$$\left(\frac{\partial^2}{\partial a_1\partial d_2}-\frac{\partial^2}{\partial a_2\partial d_1}\right)\Delta^{kl}_{00,00}\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right)$$
$$=\left(\frac{\partial^2}{\partial a_1\partial d_2}-\frac{\partial^2}{\partial a_2\partial d_1}\right)\Delta^{00}_{00,00}\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right)=0. \tag{67}$$

Therefore, from equation (43*g*), one obtains

$$\langle [A,D], a^k d^l\rangle = 0. \tag{68}$$

Now we have to compute the evaluation of the (anti)commutators between A, B, C and D on the generic elements $a^k d^l b$ and $a^k d^l c$ of the Poincaré–Birkhoff–Witt basis of $\mathcal{A}$. Let us define

$$\Delta(a^k d^l b) = \Delta(a^k d^l)(a\otimes b + b\otimes d) = \sum_{i,j,i',j'\in\{0,1\}} \beta^{kl}_{ij,i'j'}\left(\begin{pmatrix}a_1\\d_1\end{pmatrix},\begin{pmatrix}a_2\\d_2\end{pmatrix}\right) b^i c^j \otimes b^{i'} c^{j'}. \tag{69}$$

It is not difficult to obtain expressions for the polynomials $\beta^{kl}_{ij,i'j'}$ in terms of $\Delta^{kl}_{ij,i'j'}$ from the multiplication law (27). One obtains (the other possibilities are of no interest for our goal)

$$\begin{aligned}
\beta^{kl}_{00,00} &= \tfrac12 s a_1(a_2^2-d_2^2)\Delta^{kl}_{00,10} + \tfrac12 s d_2(a_1^2-d_1^2)\Delta^{kl}_{10,00}\\
\beta^{kl}_{10,00} &= d_2\Delta^{kl}_{00,00} + \tfrac12 s r a_1(a_2^2-d_2^2)\Delta^{kl}_{10,10} - \tfrac12 s^2 d_1(a_2^2-d_2^2)\Delta^{kl}_{01,10}\\
\beta^{kl}_{01,00} &= \tfrac12 r s a_1(a_2^2-d_2^2)\Delta^{kl}_{01,10} - \tfrac12 s^2 d_1(a_2^2-d_2^2)\Delta^{kl}_{10,10} + d_2(a_1^2-d_1^2)\Delta^{kl}_{11,00}\\
\beta^{kl}_{00,10} &= a_1\Delta^{kl}_{00,00} + \tfrac12 s^2 a_2(a_1^2-d_1^2)\Delta^{kl}_{10,01} - \tfrac12 r s d_2(a_1^2-d_1^2)\Delta^{kl}_{10,10}\\
\beta^{kl}_{00,01} &= \tfrac12 s a_1(a_2^2-d_2^2)\Delta^{kl}_{00,11} + \tfrac12 s^2 a_2(a_1^2-d_1^2)\Delta^{kl}_{10,10} - \tfrac12 r s d_2(a_1^2-d_1^2)\Delta^{kl}_{10,01}\\
\beta^{kl}_{10,10} &= r a_1\Delta^{kl}_{10,00} - s d_1\Delta^{kl}_{01,00} + s a_2\Delta^{kl}_{00,01} - r d_2\Delta^{kl}_{00,10}\\
\beta^{kl}_{01,01} &= \tfrac12 r s a_1(a_2^2-d_2^2)\Delta^{kl}_{01,11} - \tfrac12 s^2 d_1(a_2^2-d_2^2)\Delta^{kl}_{10,11} + \tfrac12 s^2 a_2(a_1^2-d_1^2)\Delta^{kl}_{11,10}\\
&\quad - \tfrac12 r s d_2(a_1^2-d_1^2)\Delta^{kl}_{11,01}\\
\beta^{kl}_{10,01} &= s a_2\Delta^{kl}_{00,10} - r d_2\Delta^{kl}_{00,01} + \tfrac12 r s a_1(a_2^2-d_2^2)\Delta^{kl}_{10,11} - \tfrac12 s^2 d_1(a_2^2-d_2^2)\Delta^{kl}_{01,11}\\
\beta^{kl}_{01,10} &= r a_1\Delta^{kl}_{01,00} - s d_1\Delta^{kl}_{10,00} + \tfrac12 s^2 a_2(a_1^2-d_1^2)\Delta^{kl}_{11,01} - \tfrac12 r s d_2(a_1^2-d_1^2)\Delta^{kl}_{11,10}.
\end{aligned} \tag{70}$$

The evaluations of the (anti)commutators are given by

$$\langle BC + CB, a^k d^l b\rangle = \beta^{kl}_{10,01} + \beta^{kl}_{01,10} \tag{71a}$$

$$\langle B^2, a^k d^l b\rangle = \beta^{kl}_{10,10} \qquad \langle C^2, a^k d^l b\rangle = \beta^{kl}_{01,01} \tag{71b}$$

$$\begin{aligned}\langle AB - BA, a^k d^l b\rangle &= \frac{\partial}{\partial a_1}\beta^{kl}_{00,10} - \frac{\partial}{\partial a_2}\beta^{kl}_{10,00} \\ &= s^2(\Delta^{kl}_{10,01} + \Delta^{kl}_{01,10}) - 2rs\Delta^{kl}_{10,10} + \left(1 + \frac{\partial}{\partial a_1} - \frac{\partial}{\partial a_2}\right)\Delta^{kl}_{00,00}\end{aligned} \tag{71c}$$

$$\langle AC - CA, a^k d^l b\rangle = \frac{\partial}{\partial a_1}\beta^{kl}_{00,01} - \frac{\partial}{\partial a_2}\beta^{kl}_{01,00} = 2s^2\Delta^{kl}_{10,10} - rs(\Delta^{kl}_{10,01} + \Delta^{kl}_{01,10}) \tag{71d}$$

$$\begin{aligned}\langle DB - BD, a^k d^l b\rangle &= \frac{\partial}{\partial d_1}\beta^{kl}_{00,10} - \frac{\partial}{\partial d_2}\beta^{kl}_{10,00} \\ &= 2rs\Delta^{kl}_{10,10} - s^2(\Delta^{kl}_{10,01} + \Delta^{kl}_{01,10}) + \left(-1 + \frac{\partial}{\partial d_1} - \frac{\partial}{\partial d_2}\right)\Delta^{kl}_{00,00}\end{aligned} \tag{71e}$$

$$\langle DC - CD, a^k d^l b\rangle = \frac{\partial}{\partial d_1}\beta^{kl}_{00,01} - \frac{\partial}{\partial d_2}\beta^{kl}_{01,00} = rs(\Delta^{kl}_{10,01} + \Delta^{kl}_{01,10}) - 2s^2\Delta^{kl}_{10,10} \tag{71f}$$

$$\begin{aligned}\langle AD - DA, a^k d^l b\rangle &= \left(\frac{\partial^2}{\partial a_1\partial d_2} - \frac{\partial^2}{\partial a_2\partial d_1}\right)\beta^{kl}_{00,00} \\ &= s\left(\Delta^{kl}_{10,00} - \Delta^{kl}_{00,10} - \left(\frac{\partial}{\partial a_1} + \frac{\partial}{\partial d_1}\right)\Delta^{kl}_{10,00} + \left(\frac{\partial}{\partial a_2} + \frac{\partial}{\partial d_2}\right)\Delta^{kl}_{00,10}\right)\end{aligned} \tag{71g}$$

where all polynomials $\beta^{kl}_{ij,i'j'}$ and $\Delta^{kl}_{ij,i'j'}$ and their derivatives in (71) are taken at $\binom{a_1}{d_1}, \binom{a_2}{d_2} = \binom{1}{1}, \binom{1}{1}$.

Similarly, defining

$$\Delta(a^k d^l c) = \Delta(a^k d^l)(c \otimes a + d \otimes c) = \sum_{i,j,i',j' \in \{0,1\}} \gamma^{kl}_{ij,i'j'}\left(\begin{pmatrix}a_1\\ d_1\end{pmatrix}, \begin{pmatrix}a_2\\ d_2\end{pmatrix}\right) b^i c^j \otimes b^{i'} c^{j'} \tag{72}$$

one gets

$$\begin{aligned}
\gamma^{kl}_{00,00} &= \tfrac{1}{2}sa_2(a_1^2 - d_1^2)\Delta^{kl}_{01,00} + \tfrac{1}{2}sd_1(a_2^2 - d_2^2)\Delta^{kl}_{00,01} \\
\gamma^{kl}_{10,00} &= \tfrac{1}{2}s^2a_1(a_2^2 - d_2^2)\Delta^{kl}_{01,01} - \tfrac{1}{2}rsd_1(a_2^2 - d_2^2)\Delta^{kl}_{10,01} + \tfrac{1}{2}s^2a_2(a_1^2 - d_1^2)\Delta^{kl}_{11,00} \\
\gamma^{kl}_{01,00} &= a_2\Delta^{kl}_{00,00} + \tfrac{1}{2}s^2a_1(a_2^2 - d_2^2)\Delta^{kl}_{10,01} - \tfrac{1}{2}rsd_1(a_2^2 - d_2^2)\Delta^{kl}_{01,01} \\
\gamma^{kl}_{00,10} &= \tfrac{1}{2}s^2d_1(a_2^2 - d_2^2)\Delta^{kl}_{00,11} + \tfrac{1}{2}rsa_2(a_1^2 - d_1^2)\Delta^{kl}_{01,10} - \tfrac{1}{2}s^2d_2(a_1^2 - d_1^2)\Delta^{kl}_{01,01} \\
\gamma^{kl}_{00,01} &= d_1\Delta^{kl}_{00,00} + \tfrac{1}{2}rsa_2(a_1^2 - d_1^2)\Delta^{kl}_{01,01} - \tfrac{1}{2}s^2d_2(a_1^2 - d_1^2)\Delta^{kl}_{01,10} \\
\gamma^{kl}_{10,10} &= \tfrac{1}{2}s^2a_1(a_2^2 - d_2^2)\Delta^{kl}_{01,11} - \tfrac{1}{2}rsd_1(a_2^2 - d_2^2)\Delta^{kl}_{10,11} + \tfrac{1}{2}rsa_2(a_1^2 - d_1^2)\Delta^{kl}_{11,10} \\
&\quad - \tfrac{1}{2}s^2d_2(a_1^2 - d_1^2)\Delta^{kl}_{11,01} \\
\gamma^{kl}_{01,01} &= a_1\Delta^{kl}_{10,00} - rd_1\Delta^{kl}_{01,00} + ra_2\Delta^{kl}_{00,01} - sd_2\Delta^{kl}_{00,10} \\
\gamma^{kl}_{10,01} &= \tfrac{1}{2}s^2a_1\Delta^{kl}_{01,00} - \tfrac{1}{2}rsd_1\Delta^{kl}_{10,00} + \tfrac{1}{2}rsa_2(a_1^2 - d_1^2)\Delta^{kl}_{11,01} - \tfrac{1}{2}s^2d_2(a_1^2 - d_1^2)\Delta^{kl}_{11,10} \\
\gamma^{kl}_{01,10} &= \tfrac{1}{2}rsa_2\Delta^{kl}_{00,10} - \tfrac{1}{2}s^2d_2\Delta^{kl}_{00,01} + \tfrac{1}{2}s^2a_1(a_2^2 - d_2^2)\Delta^{kl}_{10,11} - \tfrac{1}{2}rsd_1(a_2^2 - d_2^2)\Delta^{kl}_{01,11}.
\end{aligned} \tag{73}$$

Again, the evaluations of the (anti)commutators are given by

$$\langle BC + CB, a^k d^l c \rangle = \gamma^{kl}_{10,01} + \gamma^{kl}_{01,10} \tag{74a}$$

$$\langle B^2, a^k d^l c \rangle = \gamma^{kl}_{10,10} \qquad \langle C^2, a^k d^l c \rangle = \gamma^{kl}_{01,01} \tag{74b}$$

$$\langle AB - BA, a^k d^l c \rangle = \frac{\partial}{\partial a_1}\gamma^{kl}_{00,10} - \frac{\partial}{\partial a_2}\gamma^{kl}_{10,00} = rs(\Delta^{kl}_{10,01} + \Delta^{kl}_{01,10}) - 2s^2\Delta^{kl}_{01,01} \tag{74c}$$

$$\begin{aligned}\langle AC - CA, a^k d^l c \rangle &= \frac{\partial}{\partial a_1}\gamma^{kl}_{00,01} - \frac{\partial}{\partial a_2}\gamma^{kl}_{01,00} \\ &= 2rs\Delta^{kl}_{01,01} - s^2(\Delta^{kl}_{10,01} + \Delta^{kl}_{01,10}) + \left(-1 + \frac{\partial}{\partial a_1} - \frac{\partial}{\partial a_2}\right)\Delta^{kl}_{00,00}\end{aligned} \tag{74d}$$

$$\langle DB - BD, a^k d^l c \rangle = \frac{\partial}{\partial d_1}\gamma^{kl}_{00,10} - \frac{\partial}{\partial d_2}\gamma^{kl}_{10,00} = 2s^2\Delta^{kl}_{01,01} - rs(\Delta^{kl}_{10,01} + \Delta^{kl}_{01,10}) \tag{74e}$$

$$\begin{aligned}\langle DC - CD, a^k d^l c \rangle &= \frac{\partial}{\partial d_1}\gamma^{kl}_{00,01} - \frac{\partial}{\partial d_2}\gamma^{kl}_{01,00} \\ &= s^2(\Delta^{kl}_{10,01} + \Delta^{kl}_{01,10}) - 2rs\Delta^{kl}_{01,01} + \left(1 + \frac{\partial}{\partial d_1} - \frac{\partial}{\partial d_2}\right)\Delta^{kl}_{00,00}\end{aligned} \tag{74f}$$

$$\begin{aligned}\langle AD - DA, a^k d^l c \rangle &= \left(\frac{\partial^2}{\partial a_1 \partial d_2} - \frac{\partial^2}{\partial a_2 \partial d_1}\right)\gamma^{kl}_{00,00} \\ &= s\left(\Delta^{kl}_{01,00} - \Delta^{kl}_{00,01} - \left(\frac{\partial}{\partial a_1} + \frac{\partial}{\partial d_1}\right)\Delta^{kl}_{00,01} + \left(\frac{\partial}{\partial a_2} + \frac{\partial}{\partial d_2}\right)\Delta^{kl}_{01,00}\right)\end{aligned} \tag{74g}$$

(the polynomials $\gamma^{kl}_{ij,i'j'}$ and $\Delta^{kl}_{ij,i'j'}$ and their derivatives in (74) are taken at $\binom{a_1}{d_1}, \binom{a_2}{d_2} = \binom{1}{1}, \binom{1}{1}$).

Notice that all the polynomials $\Delta^{kl}_{ij,i'j'}$ and their derivatives evaluated at $\binom{a_1}{d_1}, \binom{a_2}{d_2} = \binom{1}{1}, \binom{1}{1}$, that arise in equations (71) and (74), have already been computed in the former steps of the proof (see in particular lemmas A.1, A.2, A.3 and A.4). It is therefore straightforward to obtain the evaluations of the (anti)commutators of A, B, C and D on $a^k d^l b$ and $a^k d^l c$. One finds

$$\begin{aligned}
&\langle [A, C], a^k d^l b \rangle = \langle [D, B], a^k d^l c \rangle = -\tfrac{1}{4}(q^{2k+2l} - q^{-2k-2l}) \\
&\langle [A, C], a^k d^l c \rangle = \langle [D, B], a^k d^l b \rangle = -\tfrac{1}{2} - \tfrac{1}{4}(q^{2k+2l} + q^{-2k-2l}) \\
&\langle [A, B], a^k d^l b \rangle = \langle [D, C], a^k d^l c \rangle = \tfrac{1}{2} + \tfrac{1}{4}(q^{2k+2l} + q^{-2k-2l}) \\
&\langle [A, B], a^k d^l c \rangle = \langle [D, C], a^k d^l b \rangle = \tfrac{1}{4}(q^{2k+2l} - q^{-2k-2l}) \\
&\langle [A, D], a^k d^l b \rangle = \langle \{B, C\}, a^k d^l b \rangle = \langle B^2, a^k d^l b \rangle = \langle C^2, a^k d^l b \rangle = 0 \\
&\langle [A, D], a^k d^l c \rangle = \langle \{B, C\}, a^k d^l c \rangle = \langle B^2, a^k d^l c \rangle = \langle C^2, a^k d^l c \rangle = 0.
\end{aligned} \tag{75}$$

Finally, one defines

$$\begin{aligned}\Delta(a^k d^l bc) &= \Delta(a^k d^l)\Delta(bc) = \Delta(a^k d^l)(ad \otimes bc + bc \otimes ad + rac \otimes ab - rdb \otimes dc) \\ &= \sum_{i,j,i',j' \in \{0,1\}} \mu^{kl}_{ij,i'j'}\left(\begin{pmatrix} a_1 \\ d_1 \end{pmatrix}, \begin{pmatrix} a_2 \\ d_2 \end{pmatrix}\right) b^i c^j \otimes b^{i'} c^{j'}.\end{aligned} \tag{76}$$

One gets

$$\begin{aligned}
\mu^{kl}_{00,00} &= \tfrac{1}{4}s^2 a_1 d_1 (a_2^2-d_2^2)^2 \Delta^{kl}_{00,11} + \tfrac{1}{4} rs^2 a_1 a_2 (a_1^2-d_1^2)(a_2^2-d_2^2)\Delta^{kl}_{01,10} \\
&\quad -\tfrac{1}{4} rs^2 d_1 d_2 (a_1^2-d_1^2)(a_2^2-d_2^2)\Delta^{kl}_{10,01} + \tfrac{1}{4}s^2 a_2 d_2 (a_1^2-d_1^2)^2 \Delta^{kl}_{11,00} \\
\mu^{kl}_{10,00} &= \tfrac{1}{2} s a_2 d_2 (a_1^2-d_1^2)\Delta^{kl}_{01,00} - \tfrac{1}{2} rs^2 d_1 a_2 (a_2^2-d_2^2)\Delta^{kl}_{00,10} + \tfrac{1}{2} sr^2 d_1 d_2 (a_2^2-d_2^2)\Delta^{kl}_{00,01} \\
&\quad +\tfrac{1}{4} rs^3 (a_1^2+d_1^2)(a_2^2-d_2^2)^2 \Delta^{kl}_{01,11} - \tfrac{1}{4}s^2(r^2+s^2) a_1 d_1 (a_2^2-d_2^2)^2 \Delta^{kl}_{10,11} \\
&\quad -\tfrac{1}{4} rs^3 a_1 d_2 (a_1^2-d_1^2)(a_2^2-d_2^2)\Delta^{kl}_{11,01} + \tfrac{1}{4} r^2 s^2 a_1 a_2 (a_1^2-d_1^2)(a_2^2-d_2^2)\Delta^{kl}_{11,10} \\
\mu^{kl}_{01,00} &= \tfrac{1}{2} s a_2 d_2 (a_1^2-d_1^2)\Delta^{kl}_{10,00} + \tfrac{1}{2} sr^2 a_1 a_2 (a_2^2-d_2^2)\Delta^{kl}_{00,10} - \tfrac{1}{2} rs^2 a_1 d_2 (a_2^2-d_2^2)\Delta^{kl}_{00,01} \\
&\quad -\tfrac{1}{4}s^2(r^2+s^2) a_1 d_1 (a_2^2-d_2^2)^2 \Delta^{kl}_{01,11} + \tfrac{1}{4} rs^3 (a_1^2+d_1^2)(a_2^2-d_2^2)^2 \Delta^{kl}_{10,11} \\
&\quad -\tfrac{1}{4} rs^3 d_1 a_2 (a_1^2-d_1^2)(a_2^2-d_2^2)\Delta^{kl}_{11,10} + \tfrac{1}{4} r^2 s^2 d_1 d_2 (a_1^2-d_1^2)(a_2^2-d_2^2)\Delta^{kl}_{11,01} \\
\mu^{kl}_{00,10} &= -\tfrac{1}{2} rs^2 d_1 a_2 (a_1^2-d_1^2)\Delta^{kl}_{10,00} + \tfrac{1}{2} sr^2 a_1 a_2 (a_1^2-d_1^2)\Delta^{kl}_{01,00} + \tfrac{1}{2} s a_1 d_1 (a_2^2-d_2^2)\Delta^{kl}_{00,01} \\
&\quad -\tfrac{1}{4} rs^3 a_1 d_2 (a_1^2-d_1^2)(a_2^2-d_2^2)\Delta^{kl}_{01,11} + \tfrac{1}{4} r^2 s^2 d_1 d_2 (a_1^2-d_1^2)(a_2^2-d_2^2)\Delta^{kl}_{10,11} \\
&\quad -\tfrac{1}{4}s^2(r^2+s^2) a_2 d_2 (a_1^2-d_1^2)^2 \Delta^{kl}_{11,10} + \tfrac{1}{4} rs^3 (a_1^2-d_1^2)^2 (a_2^2+d_2^2)\Delta^{kl}_{11,01} \\
\mu^{kl}_{00,01} &= \tfrac{1}{2} sr^2 d_1 d_2 (a_1^2-d_1^2)\Delta^{kl}_{10,00} - \tfrac{1}{2} rs^2 a_1 d_2 (a_1^2-d_1^2)\Delta^{kl}_{01,00} + \tfrac{1}{2} s a_1 d_1 (a_2^2-d_2^2)\Delta^{kl}_{00,10} \\
&\quad +\tfrac{1}{4} r^2 s^2 a_1 a_2 (a_1^2-d_1^2)(a_2^2-d_2^2)\Delta^{kl}_{01,11} - \tfrac{1}{4} rs^3 d_1 a_2 (a_1^2-d_1^2)(a_2^2-d_2^2)\Delta^{kl}_{10,11} \\
&\quad -\tfrac{1}{2}s^2(r^2+s^2) a_2 d_2 (a_1^2-d_1^2)^2 \Delta^{kl}_{11,01} + \tfrac{1}{4} rs^3 (a_1^2-d_1^2)^2 (a_2^2+d_2^2)\Delta^{kl}_{11,10} \\
\mu^{kl}_{10,10} &= \tfrac{1}{2} rs a_1 a_2 (a_1^2-d_1^2)\Delta^{kl}_{11,00} - \tfrac{1}{2} rs d_1 d_2 (a_2^2-d_2^2)\Delta^{kl}_{00,11} + rs^2 (a_1^2 a_2^2 - d_1^2 d_2^2)\Delta^{kl}_{01,01} \\
&\quad -\tfrac{1}{2} s(r^2+s^2) a_1 d_1 (a_2^2-d_2^2)\Delta^{kl}_{10,01} - \tfrac{1}{2} s(r^2+s^2) a_2 d_2 (a_1^2-d_1^2)\Delta^{kl}_{01,10} \\
\mu^{kl}_{01,01} &= \tfrac{1}{2} rs a_1 a_2 (a_2^2-d_2^2)\Delta^{kl}_{00,11} - \tfrac{1}{2} rs d_1 d_2 (a_1^2-d_1^2)\Delta^{kl}_{11,00} + rs (a_1^2 a_2^2 - d_1^2 d_2^2)\Delta^{kl}_{10,10} \\
&\quad -\tfrac{1}{2} s(r^2+s^2) a_1 d_1 (a_2^2-d_2^2)\Delta^{kl}_{01,10} - \tfrac{1}{2} s(r^2+s^2) a_2 d_2 (a_1^2-d_1^2)\Delta^{kl}_{10,01} \\
\mu^{kl}_{10,01} &= -r d_1 d_2 \Delta^{kl}_{00,00} + \tfrac{1}{4} rs^2 a_1 a_2 (a_1^2-d_1^2)(a_2^2-d_2^2)\Delta^{kl}_{11,11} + rs^2 (a_1^2 a_2^2 - d_1^2 d_2^2)\Delta^{kl}_{01,10} \\
&\quad -\tfrac{1}{2} s(r^2+s^2) a_1 d_1 (a_2^2-d_2^2)\Delta^{kl}_{10,10} - \tfrac{1}{2} s(r^2+s^2) a_2 d_2 (a_1^2-d_1^2)\Delta^{kl}_{01,01} \\
\mu^{kl}_{01,10} &= r a_1 a_2 \Delta^{kl}_{00,00} - \tfrac{1}{4} rs^2 d_1 d_2 (a_1^2-d_1^2)(a_2^2-d_2^2)\Delta^{kl}_{11,11} + rs^2 (a_1^2 a_2^2 - d_1^2 d_2^2)\Delta^{kl}_{10,01} \\
&\quad -\tfrac{1}{2} s(r^2+s^2) a_2 d_2 (a_1^2-d_1^2)\Delta^{kl}_{10,10} - \tfrac{1}{2} s(r^2+s^2) a_1 d_1 (a_2^2-d_2^2)\Delta^{kl}_{01,01}.
\end{aligned} \tag{77}$$

Once again, the evaluations of the (anti)commutators are given by

$$\langle BC + CB, a^k d^l bc\rangle = \mu^{kl}_{10,01} + \mu^{kl}_{01,10} \tag{78a}$$

$$\langle B^2, a^k d^l bc\rangle = \mu^{kl}_{10,10} \qquad \langle C^2, a^k d^l bc\rangle = \mu^{kl}_{01,01} \tag{78b}$$

$$\begin{aligned}\langle AB - BA, a^k d^l bc\rangle &= \frac{\partial}{\partial a_1}\mu^{kl}_{00,10} - \frac{\partial}{\partial a_2}\mu^{kl}_{10,00} \\ &= -rs^2\Delta^{kl}_{10,00} + sr^2\Delta^{kl}_{01,00} - rs^2\Delta^{kl}_{00,10} + sr^2\Delta^{kl}_{00,01}\end{aligned} \tag{78c}$$

$$\begin{aligned}\langle AC - CA, a^k d^l bc\rangle &= \frac{\partial}{\partial a_1}\mu^{kl}_{00,01} - \frac{\partial}{\partial a_2}\mu^{kl}_{01,00} \\ &= sr^2\Delta^{kl}_{10,00} - rs^2\Delta^{kl}_{01,00} + sr^2\Delta^{kl}_{00,10} - rs^2\Delta^{kl}_{00,01}\end{aligned} \tag{78d}$$

$$\langle DB - BD, a^k d^l c\rangle = \frac{\partial}{\partial d_1}\mu^{kl}_{00,10} - \frac{\partial}{\partial d_2}\mu^{kl}_{10,00}$$
$$= rs^2\Delta^{kl}_{10,00} - sr^2\Delta^{kl}_{01,00} + rs^2\Delta^{kl}_{00,10} - sr^2\Delta^{kl}_{00,01} \qquad (78e)$$
$$\langle DC - CD, a^k d^l bc\rangle = \frac{\partial}{\partial d_1}\mu^{kl}_{00,01} - \frac{\partial}{\partial d_2}\mu^{kl}_{01,00}$$
$$= -sr^2\Delta^{kl}_{10,00} + rs^2\Delta^{kl}_{01,00} - sr^2\Delta^{kl}_{00,10} + rs^2\Delta^{kl}_{00,01} \qquad (78f)$$
$$\langle AD - DA, a^k d^l bc\rangle = \left(\frac{\partial^2}{\partial a_1 \partial d_2} - \frac{\partial^2}{\partial a_2 \partial d_1}\right)\mu^{kl}_{00,00} \qquad (78g)$$

(the polynomials $\mu^{kl}_{ij,i'j'}$ and $\Delta^{kl}_{ij,i'j'}$ and their derivatives in (78) are taken at $\binom{a_1}{d_1}, \binom{a_2}{d_2} = \binom{1}{1}, \binom{1}{1}$).

Expressions (78*a*) and (78*b*) are obviously vanishing while expressions (78*c*–*f*) are zero thanks to lemma A.2. Finally,

$$\frac{\partial^2}{\partial a_1 \partial d_2}\mu^{kl}_{00,00}\left(\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right) = \frac{\partial^2}{\partial a_2 \partial d_1}\mu^{kl}_{00,00}\left(\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right) = 4r(\Delta^{kl}_{10,01} - \Delta^{kl}_{01,10})$$

so that expression (78*g*) is also vanishing. Therefore one has

$$\langle [A,B], a^k d^l bc\rangle = \langle [A,C], a^k d^l bc\rangle = \langle [D,B], a^k d^l bc\rangle = \langle [D,C], a^k d^l bc\rangle = 0$$
$$\langle [A,D], a^k d^l bc\rangle = \langle \{B,C\}, a^k d^l bc\rangle = \langle B^2, a^k d^l bc\rangle = \langle C^2, a^k d^l bc\rangle = 0. \qquad (79)$$

The last step of the proof consists of interpreting the formulae (53), (54), (59), (68), (75) and (79), that are evaluations of the (anti)commutators of the elements A, B, C and D of $\mathcal{U}$, onto the generic elements of the Poincaré–Birkhoff–Witt basis of $\mathcal{A}$, as abstract formulae defining the algebra given in theorem 3. One has

$$\langle (A+D)^n, a^k d^l\rangle = \langle \otimes_n (A+D), \Delta^{(n)}(a^k d^l)\rangle. \qquad (80)$$

The generalization of formula (41) for the n-fold co-product reads as

$$\Delta^{(n)}(a^k d^l) = \sum_{i_1,j_1,\ldots,i_n,j_n \in \{0,1\}} \Delta^{(n)kl}_{i_1 j_1,\ldots,i_n j_n}\left(\begin{pmatrix}a_1\\d_1\end{pmatrix}\ldots\begin{pmatrix}a_n\\d_n\end{pmatrix}\right) b^{i_1}c^{j_1}\otimes\cdots\otimes b^{i_n}c^{j_n} \qquad (81)$$

where $a_i = 1\otimes\cdots\otimes 1\otimes a\otimes 1\cdots\otimes 1$ and a stand in place of i of the tensor product, with a similar definition for d_i. Thus one has

$$\langle \otimes_n (A+D), \Delta^{(n)}(a^k d^l)\rangle = \left\langle \otimes_n (A+D), \Delta^{(n)kl}_{00,\ldots,00}\left(\begin{pmatrix}a_1\\d_1\end{pmatrix}\ldots\begin{pmatrix}a_n\\d_n\end{pmatrix}\right)\right\rangle. \qquad (82)$$

Now the main observation is that the terms in (81) coming from b^2 or c^2 are cancelled when evaluated on $A+D$ since $\langle A+D, a^k d^l b^2\rangle = \langle A+D, a^k d^l c^2\rangle = \langle A+D, \frac{1}{2}sa^k d^l(a^2-d^2)\rangle = 0$. It follows that the only relevant term of $\Delta^{(n)kl}_{00,\ldots,00}\left(\binom{a_1}{d_1}\ldots\binom{a_n}{d_n}\right)$ is $a_1^k d_1^l \ldots a_n^k d_n^l = a^k d^l \otimes\cdots\otimes a^k d^l$. Therefore

$$\langle (A+D)^n, a^k d^l\rangle = (k+l)^n \qquad (83)$$

from which we easily deduce

$$\langle q^{A+D}, a^k d^l\rangle = \langle K, a^k d^l\rangle = q^{k+l}. \qquad (84)$$

Moreover, one has from equations (69), (70), (72) and (73) and previous results (note the shift in the exponential !) that

$$\langle q^{A+D-1}B, a^k d^l b\rangle = \langle q^{A+D-1}C, a^k d^l c\rangle = q^{k+l}. \qquad (85)$$

Then by comparing equations (53), (54), (59), (68), (75) and (79) with the formulae (84) and (85), theorem 3 immediately follows. □

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